SEMICLASSICAL RESONANCES FOR A TWO-LEVEL SCHRÖDINGER OPERATOR WITH A CONICAL INTERSECTION

S. FUJIIÉ, C. LASSER, AND L. NÉDÉLEC

Abstract. We study the resonant set of a two-level Schrödinger operator with a linear conical intersection. This model operator can be decomposed into a direct sum of first order systems on the real half-line. For these ordinary differential systems we locally construct exact WKB solutions, which are connected to global solutions, amongst which are resonant states. The main results are a generalized Bohr-Sommerfeld quantization condition and an asymptotic description of the set of resonances as a distorted lattice.

1. Introduction

This paper is devoted to the semiclassical distribution of resonances of the two-dimensional and two-level Schrödinger operator

$$P = -h^2\Delta_x + V(x) = -h^2\Delta_x + \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}.$$

Such two-level operators appear naturally in the study of molecular spectra: if the positions of a molecule's nuclei are denoted by $x \in \mathbb{R}^n$, and $0 < h \ll 1$ is Planck's constant devided by the square-root of the nuclear mass, then a full molecular Hamiltonian reads as $H_{\mathrm{mol}} = -h^2\Delta_x + H_{\mathrm{el}}(x)$. For every nucleonic position x, the electronic Hamiltonian $H_{\mathrm{el}}(x)$ is an operator on the electronic degrees of freedom. The full operator H_{mol} acts on nucleonic *and* electronic degrees of freedom, that is on wave functions in $L^2(\mathbb{R}^N, \mathbb{C})$, where $N = 3\cdot$(number of nuclei+electrons) is a notoriously large number. If one considers two eigenvalues of the electronic Hamiltonian $H_{\mathrm{el}}(x)$, which are well separated from the rest of the electronic spectrum $\sigma(H_{\mathrm{el}}(x))$ uniformly for all x, then Born-Oppenheimer approximation allows to reduce the study of the full molecular problem to the case of two-level systems acting solely on the nucleonic degrees of freedom. The justification of this approximation can be found in [8, 22] for the time-independent and in [16, 18, 32] for the time-dependent case. For considerations from the point of view of theoretical chemistry, we also refer to [6].

Chapter 2 in the monograph [17] gives the standard classification of matrix Schrödinger operators with eigenvalue crossings of minimal multiplicity, where our model operator P appears as the normal form for a codimension two crossing: the real symmetric potential matrix $V(x) = V(x_1, x_2)$ depends smoothly on the two real parameters x_1 and x_2. It has two eigenvalues

$$\pm\sqrt{x_1^2 + x_2^2} = \pm|x|,$$

which coincide on the codimension two manifold $\{x = 0\} \subset \mathbb{R}^2$. The graph of the mapping $x \mapsto \pm|x|$ shows two cones intersecting at the origin, which explains the term conical

Supported by MSRI and JSPS. KAKENHI15540149.
Supported by DFG priority program 1095.
Supported in part by the FIM of ETHZ.

intersection, which is used in the chemical physics' literature (see for example [5, 10, 33]). Such a degeneracy, geared by two parameters, is generic in the sense, that it cannot be removed by symmetry preserving perturbations. The matrix $V(x)$ is in essence Rellich's celebrated example of a smooth matrix, which is not smoothly diagonalizable [30].

If the crossing were not present, one might decouple the system and study the two scalar Hamiltonians

$$P^{\pm} = -h^2\Delta_x \pm |x|\,,$$

see [25, 26, 4]. P^+ is a Schrödinger operator with confining potential, and one has pure point spectrum only (see Appendix B). The lower level operator P^-, however, has a linearly decreasing negative potential, and one sees by a Mourre-type argument, that P^- is of purely continuous spectrum. The full operator P inherits the continuous spectrum of P^-, while spectrally echoing the discrete spectrum of P^+ with resonances close to the real axis.

In the physical literature, zero energy wave functions close to a conical level crossing [2, 3] have been studied, and conical intersections have also been addressed as generators of resonances [5].

A first mathematical proof of existence of resonances for the matrix operator P has been given by one of the authors [27]. She crucially used that P is unitarily equivalent to the direct sum of ordinary differential operators

$$\bigoplus_{\nu\in h(\mathbb{Z}+\frac{1}{2})} P_\nu, \quad P_\nu = \begin{pmatrix} r^2 - hD_r & \nu/r \\ \nu/r & r^2 + hD_r \end{pmatrix} \tag{1}$$

where we use the notation $D_r = -i\partial/\partial r$. This decomposition is achieved by a h-Fourier transformation, a change to polar coordinates, a Prüfer transformation, and a h-Fourier series ansatz in the angular variable. In particular, r is the length $|\xi|$ of the dual variables $\xi = (\xi_1, \xi_2)$ of (x_1, x_2) and ν is the quantum angular momentum.

The energy surfaces of the Hamiltonian flow associated with the eigenvalues $r^2 \pm \sqrt{\rho^2 + \nu^2/r^2}$ of the symbol $p_\nu(r, \rho)$ of P_ν are

$$\left\{(r,\rho) \in \mathbb{R}^+ \times \mathbb{R};\ \rho^2 = (E - r^2)^2 - \nu^2/r^2\right\}.$$

If $\nu^2 < 4E^3/27$, then the function $(E - r^2)^2 - \nu^2/r^2$ has three distinct positive zeros $0 < r_0 < r_1 < r_2$, see Appendix A. The Hamiltonian flow consists of two components, a periodic flow passing through $(r_0, 0)$ and $(r_1, 0)$ and a flow coming from infinity and going away to infinity passing through $(r_2, 0)$. Let $E > 0$ be positive. Then r_0 tends to 0 while r_1 and r_2 tend to $\sqrt{E}$ as $h \to 0$: the periodic and the unbounded component approach each other in the semiclassical limit.

The resonances of the operator P are defined as the eigenvalues of the complex scaled Hamiltonian

$$P_\theta = -h^2 \mathrm{e}^{-2i\theta}\Delta + \mathrm{e}^{i\theta} V(x)\,,$$

which is a non-selfadjoint operator with discrete spectrum independent of the dilation parameter $\theta \in]0, \frac{\pi}{3}[$ (see for example [1, 20, 31, 34] for general theory of resonances and [27] for the model operator P). In terms of the reduced operators P_ν, resonances are characterized as follows:

Proposition 1.1 *$E \in \mathbb{C}$ is a resonance of the operator $P = -h^2\Delta + V$ if and only if there exist $\nu \in h(\mathbb{N} - \frac{1}{2}) = \{\frac{h}{2}, \frac{3h}{2}, \ldots\}$ and a non-trivial solution w to the equation*

$$P_\nu\, w = E\, w \tag{2}$$

satisfying

$$\lim_{r\to 0+} w(r) = 0, \quad r^2 w(e^{-i\theta}r),\ w'(e^{-i\theta}r) \in L^2(\mathbb{R}^+, \mathbb{C}^2), \tag{3}$$

for some $\theta \in]0, \pi/3[$.

The structure of the resonant set can now be analysed by constructing a solution u of the undilated system $P_\nu u = Eu$ for $E \in \mathbb{C}$, which vanishes at $r = 0$ and is incoming at $r = +\infty$ (see Proposition 4.1).

Let $S_{01}(E, h)$ be the action integral for the periodic flow defined by

$$S_{01}(E,h) = \int_{r_0}^{r_1} \frac{\sqrt{\nu^2 - r^2(E-r^2)^2}}{r} dr,$$

where the square root is defined to be positive when E is positive. It is extended analytically into a complex neighborhood of $\{E > 0\}$, and if $|E|$ is bounded from below, the asymptotic behavior of $S_{01}(E, h)$ is given by

$$S_{01}(E,h) = \tfrac{2}{3} i E^{3/2} + \tfrac{\pi}{2} i \tilde\nu h + O(h^2 |\ln h|) \qquad (h \to 0). \tag{4}$$

This yields the following Bohr-Sommerfeld type quantization condition for each angular momentum $\tilde\nu \in \mathbb{N} - \frac{1}{2}$:

Theorem 1.2 *Let $E_0 > 0$ and $\tilde\nu \in \mathbb{N} - \frac{1}{2} = \{\frac{1}{2}, \frac{3}{2}, \dots\}$ be given. Then there exist $\epsilon > 0$, $h_0 > 0$ and a function $\delta(E,h) : \{(E,h) \in \mathbb{C} \times \mathbb{R}_+;\ |E - E_0| < \epsilon,\ 0 < h < h_0\} \to \mathbb{C}$ with $\delta(E,h) \to 0$ uniformly in E as $h \to 0$, such that equation (2) has a non-trivial solution satisfying (3) if and only if (E,h) satisfies the following quantization condition:*

$$\sqrt{\frac{\pi h}{2}}\, \tilde\nu\, e^{-i\pi/4}\, E^{-3/4}\, e^{2S_{01}(E,h)/h} + 1 = \delta(E,h). \tag{5}$$

Combining this result with Proposition 1.1 and (4), one obtains the following theorem about the distribution of resonances. Here, we take $\lambda = E^{3/2}$ as spectral parameter and look for resonances in $\{\lambda \in \mathbb{C};\ a < \operatorname{Re}\lambda < b,\ \operatorname{Im}\lambda < 0,\ \operatorname{Im}\lambda = o(1) \text{ as } h \to 0\}$ for arbitrarily fixed positive numbers $a, b > 0$. For each $\tilde\nu \in \mathbb{N} - \frac{1}{2}$, we define

$$\Gamma_{\tilde\nu}(h) = \left\{\lambda \in \mathbb{C};\ \lambda = \lambda_{k\tilde\nu} h - i\,\tfrac{3}{8}\left(h \ln \tfrac{1}{h} - h \ln \frac{\pi\tilde\nu^2}{2\lambda_{k\tilde\nu} h}\right),\ k \in \mathbb{Z} \text{ s.t. } a < \lambda_{k\tilde\nu} h < b\right\},$$

where $\lambda_{k\tilde\nu} = \frac{3\pi}{16}(8k - 4\tilde\nu + 5)$.

Theorem 1.3 *Let $a, b > 0$. Then, for any $N \in \mathbb{N}$ there exists $h_0 > 0$ such that for any $0 < h < h_0$ and $\lambda \in \bigcup_{\tilde\nu \le N} \Gamma_{\tilde\nu}(h)$ there is a resonance E of the operator P with $\lambda - E^{3/2} = o(h)$ uniformly for all $\lambda \in \bigcup_{\tilde\nu \le N} \Gamma_{\tilde\nu}(h)$ as $h \to 0$.*

Remark 1.4 The integer parameters $k \in \mathbb{Z}$ should be large of $O(h^{-1})$ since $a < \lambda_{k\tilde\nu} h$. Hence, the second term of the imaginary part of $\lambda \in \Gamma_{\tilde\nu}(h)$ is of $O(h)$ and smaller than the first term. Thus, $\Gamma_{\tilde\nu}(h)$ is an almost horizontal sequence of complex points in the λ-plane, and $\bigcup_{\tilde\nu \le N} \Gamma_{\tilde\nu}(h)$ is a lattice which consists of N horizontal sequences. Theorem 1.3 means that for a fixed positive interval $[a, b]$, we can find as many horizontal sequences as we want for sufficiently small h, which are close to resonances of the operator P. $\diamond$

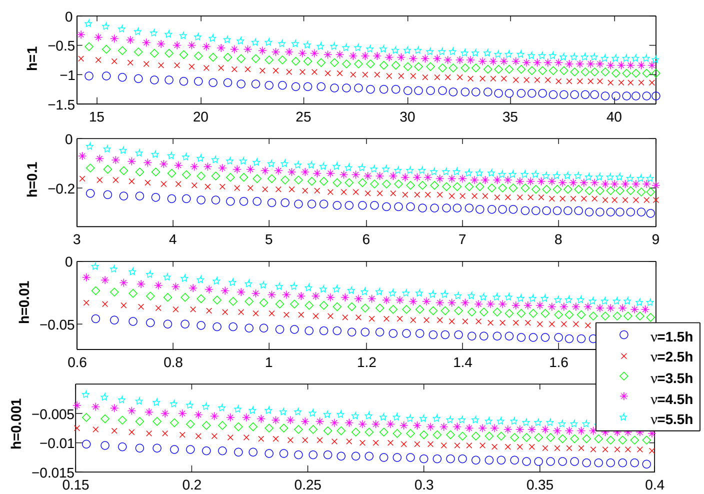

FIGURE 1. Resonances of the operator $P = -h^2\Delta + V(x)$. The parameter k lies in $\{11, 12, \ldots, 60\}$, while ν is chosen in $\{1.5h, 2.5h, \ldots, 5.5h\}$. The semiclassical parameter h varies from 10^{-3} to 1.

The plots in Figure 1 illustrate the distorted lattice of resonances given by Theorem 1.3. The larger the angular momentum number $\tilde{\nu} \in \mathbb{N} - \frac{1}{2}$, the closer the resonance to the real axis. That is, larger angular momentum numbers are associated with longer life time of the corresponding resonant states. Studies of the dynamical properties [11, 23, 24] of the model operator P complement this observation by a quick heuristic argument: The symbols $p^{\pm}(x,\xi) = |\xi|^2 \pm |x|$ of the one-level operators $P^{\pm} = -h^2\Delta \pm |x|$ induce Hamiltonian systems

$$\dot{x} = 2\xi\,, \qquad \dot{\xi} = \mp \frac{x}{|x|}$$

with central field. Such systems conserve angular momentum $x \wedge \xi = x_1\xi_2 - x_2\xi_1$. Hence, only trajectories with small angular momentum come close to the crossing manifold $\{x = 0\}$. Moreover, angular momentum encodes closeness of trajectories to the crossing. Straightforward arguments yield that the Hamiltonian system associated with p^+ has constraint motion including periodic orbits, while the motion corresponding to p^- is unbounded. Hence, on a heuristic level, the classical dynamics of the decoupled systems reflect the structure of the full operator's set of resonances: a high angular momentum number $\widetilde{\nu}$ of a resonance mirrors a periodic orbit of the upper level with high angular momentum. Such orbits in turn imply existence of localized quasimodes and long-living

resonant states. On the other hand, small angular momentum numbers $\widetilde{\nu}$ correspond to orbits close to the crossing manifold. Nearby the crossing, non-adiabatic transitions to the unbounded motion of the minus-system are possible. Hence, in this regime shorter life-times and resonances far away from the real axis have to be expected. This heuristic point of view is in wonderful agreement with the derivation of the Bohr-Sommerfeld conditions (5), and the prefactor

$$\sqrt{\frac{\pi h}{2}}\,\tilde{\nu}\,e^{-i\pi/4}\,E^{-3/4}$$

can be seen as some type of Landau-Zener probabilty for *not* performing a non-adiabatic transition near the crossing. Hence, the conditions (5) is an interpretable extension of quantization conditions for scalar Schrödinger operators to the case of a matrix-valued operator with crossing eigenvalues. For a study relating the resonant set of the matrix operator P with the spectrum of the scalar operator P^+ we refer to Appendix B.

The method of proving Theorem 1.2 and 1.3 is similar to [13], while the problem resembles in some part the double well problem [19]. We construct exact solutions of the equations $P_\nu u = Eu$, $\nu \in h(\mathbb{N} - \frac{1}{2})$, which are exact WKB solutions in local domains of the complex plane. Connecting these solutions from domain to domain, we get global solutions. More precisely, we construct an exact solution which vanishes at the origin, and represent it, after several connection procedures, as a linear combination of Jost solutions, which are defined at infinity. The quantization condition of resonances will be given as the condition that the connection coefficient of the outgoing Jost solution vanishes (Proposition 4.1).

The proof of our main results, Theorem 1.2 and Theorem 1.3, involves several different tools and proceeds as follows: In §2, we reduce the study of the full operator P to that of the ordinary differential operators P_ν, $\nu \in h(\mathbb{N} - \frac{1}{2})$, and prove Proposition 1.1. In §3, we extend the theory of exact WKB analysis for one-dimensional scalar Schrödinger operators used in [15] to a class of Schrödinger systems. This exact WKB theory works in the generic situation, where the operator is without singularity, and turning points are "sufficiently well-separated" from each other with respect to h. Our systems, however, have the origin as a regular singular point. Moreover, the first turning point r_0 tends to this singularity, while the other two turning points r_1 and r_2 approach each other at $\sqrt{E}$ as $h \to 0$. §4 outlines the strategy to obtain a global solution under these circumstances. In §5, we define Jost solutions and represent them as exact WKB solutions. In §6, we construct an exact WKB solution vanishing at the origin, facing the same kind of difficulty as in [14]. The bad error estimates in Theorem 1.2 and Theorem 1.3 come only from here. §7 is devoted to the connection formula at $\sqrt{E}$. Here, we reduce the operator P_ν to a well-studied microlocal normal form. In §8, we compute the quantization condition (Theorem 1.2) from the connection formulae obtained in the preceding sections. This condition is given in the form of Bohr-Sommerfeld using the action $S_{01}(E,h)$ between r_0 and r_1. Analysing the asymptotic behavior of the action $S_{01}(E,h)$, we finally get the semiclassical distribution of resonances (Theorem 1.3).

Acknowledgments. Part of this work has been carried out while the authors have been members of the program on semiclassical analysis at the MSRI, Berkeley. We thank Maciej Zworski for valuable discussions.

Contents

2. Reduction to the first order ordinary differential system

The aim of this section is twofold: first, we define the resonances of

$$P = -h^2\Delta + V(x), \quad V(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix} \tag{6}$$

as the eigenvalues of its complex scaled counterpart

$$P_\theta = -e^{-2i\,\theta}\, h^2\Delta + e^{i\,\theta}\, V(x)\,, \qquad \theta \in]0, \tfrac{\pi}{3}[\,.$$

Then, we reduce the study of the two-dimensional operators P and P_θ to countably many one-dimensional operators $\{P_\nu\}_{\nu\in h(1/2+\mathbb{Z})}$,

$$P_\nu = \begin{pmatrix} r^2 - hD_r & \nu/r \\ \nu/r & r^2 + hD_r \end{pmatrix}$$

and $\{P_{\theta,\nu}\}_{\nu\in h(1/2+\mathbb{Z})}$

$$P_{\theta,\nu} = \begin{pmatrix} e^{-2i\theta}r^2 - e^{i\theta}hD_r & e^{i\theta}\nu/r \\ e^{i\theta}\nu/r & e^{-2i\theta}r^2 + e^{i\theta}hD_r \end{pmatrix},$$

where D_r stands for $-i\partial/\partial r$. Most of the material presented here can be found in [27]. We give a self-contained account of it for the convenience of the reader.

Lemma 2.1 *Let $\theta \in]0, \frac{\pi}{3}[$. The operator P_θ with domain*

$$\mathcal{D} = \left\{u \in H^2(\mathbb{R}^2, \mathbb{C}^2)\,;\ |x|u \in L^2(\mathbb{R}^2, \mathbb{C}^2)\right\}$$

is closed and has purely discrete spectrum, $\sigma(P_\theta) = \sigma_{\rm disc}(P_\theta)$. The spectrum is independent of the scaling parameter $\theta \in]0, \frac{\pi}{3}[$ in the sense that

$$\sigma_{\rm disc}(P_\theta) = \sigma_{\rm disc}(P_{\theta'}) \qquad (0 < \theta < \theta' < \tfrac{\pi}{3})\,.$$

Proof: We split the proof in three steps.

Closedness. Let $\alpha \in \mathbb{C}$ with $\operatorname{Im}\alpha \neq 0$ and $T_\alpha := -h^2\Delta + \alpha V(x)$. Then, the following quadratic estimate holds: there exist positive constants $c_\alpha, b_\alpha > 0$ such that

$$\|T_\alpha u\|_{L^2}^2 + c_\alpha \|u\|_{L^2}^2 \geq b_\alpha \left(\| - h^2\Delta u\|_{L^2}^2 + \| \, |x| u\|_{L^2}^2\right) \tag{7}$$

for all $u \in C_c^\infty(\mathbb{R}^2, \mathbb{C}^2)$. Indeed, denoting $D = -i\nabla_x$,

$$T_{\bar\alpha}\, T_\alpha = (hD)^4 + \operatorname{Im}(\alpha)\, V(hD) + \operatorname{Re}(\alpha)\left(-h^2\Delta V(x) - V(x)h^2\Delta\right) + |\alpha|^2 |x|^2 \,,$$

where one uses $i[-h^2\Delta, V(x)] = V(hD)$. Since $V(x)^2 = |x|^2$, one gets

$$0 \leq \left(-h^2|\alpha|^{-\frac{1}{2}}\Delta \pm |\alpha|^{\frac{1}{2}} V(x)\right)^2 = |\alpha|^{-1}(hD)^4 \pm (-h^2\Delta)V(x) \pm V(x)(-h^2\Delta) + |\alpha|\,|x|^2$$

and

$$\operatorname{Re}(\alpha)\left(-h^2\Delta V(x) - V(x)h^2\Delta\right) \geq -|\operatorname{Re}(\alpha)|\,|\alpha|^{-1}\left((hD)^4 + |\alpha|^2|x|^2\right) .$$

Since $V(\cdot)$ behaves like $\pm|\cdot|$, one obtains

$$\begin{aligned} T_{\bar\alpha}\, T_\alpha &\geq (1 - |\operatorname{Re}(\alpha)|\,|\alpha|^{-1})\left((hD)^4 + |\alpha|^2|x|^2\right) + \operatorname{Im}(\alpha) V(hD) \\ &\geq b_\alpha\left((hD)^4 + |x|^2\right) - c_\alpha \end{aligned}$$

and the desired estimate (7). Literally the same arguments as in the proof of Theorem II.3 in [20] conclude the proof, that T_α is a closed operator on $\mathcal{D}$.

Discrete spectrum. Let $\theta \in]0, \frac{1}{3}[$. Lemma 3.1(ii) in [27] proves, that the mapping $P_\theta - z_\theta : \mathcal{D} \to L^2(\mathbb{R}^2, \mathbb{C}^2)$ is bijective for all $z_\theta \in \mathbb{C}$ with $\operatorname{Im}(z_\theta e^{-i\theta}) > 0$. By Rellich's compactness theorem, the embedding $\mathcal{D} \to L^2(\mathbb{R}^2, \mathbb{C}^2)$ is compact. Hence, $(P_\theta - z_\theta)^{-1}$ is compact. Let $z_\theta \in \mathbb{C}$ with $\operatorname{Im}(z_\theta e^{-i\theta}) > 0$. For $z \in \mathbb{C}$ one writes

$$(P_\theta - z)(P_\theta - z_\theta)^{-1} = I + (z_\theta - z)(P_\theta - z_\theta)^{-1} =: I + K_\theta(z) \,,$$

where $\{K_\theta(z)\}_{z\in\mathbb{C}}$ is an analytic family of compact operators. For z sufficiently close to z_θ, one has $\|K_\theta(z)\| < 1$. Then, by the analytic Fredholm theorem, $z \mapsto (I + K_\theta(z))^{-1}$ and $z \mapsto (P_\theta - z)^{-1}$ are meromorphic in $\mathbb{C}$, and the residues at the poles are finite rank operators. Hence, $\sigma(P_\theta) = \sigma_{\mathrm{disc}}(P_\theta)$.

Independence of θ. For $\phi \in S = \left\{\phi \in \mathbb{C};\ \operatorname{Im}\phi \in]0, \frac{\pi}{3}[\right\}$ one defines

$$H_\phi := -e^{-2\phi} h^2\Delta + e^{\phi}\, V(x)$$

as a closed operator with domain $\mathcal{D}$. Since $H_\phi = U_{\operatorname{Re}\phi} P_\theta\, U_{-\operatorname{Re}\phi}$ with $\theta = \operatorname{Im}\phi$ and $U_{\operatorname{Re}\phi} u(x) = e^{\operatorname{Re}\phi} u(e^{\operatorname{Re}\phi} x)$ the unitary scaling, one has $\sigma(H_\phi) = \sigma(P_\theta)$. Since $\{H_\phi\}_{\phi\in S}$ is a holomorphic family of type (A), eigenvalues E_ϕ of H_ϕ depend analytically on ϕ, see VII.§3 in [21]. By unitarity, $\phi \mapsto E_\phi$ is constant if $\operatorname{Im}\phi$ is constant. Hence, the eigenvalues of H_ϕ and P_θ are independent of ϕ and θ, respectively. □

We note, that the constants $b_\alpha, c_\alpha > 0$ used in the preceding proof approach zero as $\operatorname{Im}\alpha$ tends to zero. Hence, the above quadratic estimate does not yield closedness of the undilated operator $P = -h^2\Delta + V(x)$ on the domain $\mathcal{D}$. However, a straight forward adaption of the Faris-Lavine Theorem proves essential self-adjointness of P on $\mathcal{D}$.

Definition 2.2 The eigenvalues of the dilated operator $P_\theta = -e^{-2i\,\theta}\, h^2\Delta + e^{i\,\theta}\, V(x)$, $\theta \in]0, \frac{\pi}{3}[$, with domain $\mathcal{D} = \left\{u \in H^2(\mathbb{R}^2, \mathbb{C}^2)\,;\ |x| u \in L^2(\mathbb{R}^2, \mathbb{C}^2)\right\}$ are called the *resonances* of the operator $P = -h^2\Delta + V(x)$.

The choice of dilation angle $\theta \in]0, \frac{\pi}{3}[$ is crucial: under $x \mapsto -x$ the dilated operator $P_{\pi/3} = -e^{-2i\pi/3}h^2\Delta + e^{i\pi/3}V(x)$ is unitarily equivalent to the original operator P, which has no discrete spectrum, see Proposition 5.2.

Lemma 2.3 *Let us fix $\theta \in]0, \frac{\pi}{3}[$. Then $E \in \mathbb{C}$ is a resonance of P if and only if there exists $\nu \in h(\frac{1}{2} + \mathbb{Z})$ such that $E \in \mathbb{C}$ is an eigenvalue of the operator $P_{\theta,\nu}$ with domain*

$$\tilde{\mathcal{D}}_\nu = \left\{ w \in H^1(\mathbb{R}^+, \mathbb{C}^2);\ r^{-1}w,\ r^2 w \in L^2(\mathbb{R}^+, \mathbb{C}^2) \right\}$$

if $\nu \neq \frac{h}{2}$, and

$$\tilde{\mathcal{D}}_{\pm\frac{h}{2}} = \left\{ w \in L^2(\mathbb{R}^+, \mathbb{C}^2);\ (-D_r w_1 \pm (2r)^{-1} w_2),\ (D_r w_2 \pm (2r)^{-1} w_1),\ r^2 w \in L^2(\mathbb{R}^+, \mathbb{C}^2) \right\}.$$

Proof: Again we proceed in several steps.

Fourier transformation and polar coordinates. Let $\hat{u}(\xi) \equiv (\mathcal{F}_h u)(\xi)$ be the semiclassical Fourier transform of u,

$$\hat{u}(\xi) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{-ix\cdot\xi/h} u(x) dx.$$

Then the dilated equation

$$P_\theta u_\theta = E u_\theta \tag{8}$$

becomes

$$\left\{ e^{-2i\theta}|\xi|^2 - he^{i\theta} \begin{pmatrix} D_{\xi_1} & D_{\xi_2} \\ D_{\xi_2} & -D_{\xi_1} \end{pmatrix} - E \right\} \hat{u}_\theta = 0.$$

Switching to polar coordinates $(\xi_1, \xi_2) = r(\cos\phi, \sin\phi)$, $r \in \mathbb{R}^+$, $\phi \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ one gets

$$\left\{ e^{-2i\theta} r^2 - hA(\phi)e^{i\theta} D_r - \frac{h}{r} A'(\phi) e^{i\theta} D_\phi - E \right\} \hat{u}_\theta = 0$$

with

$$A(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

Since the volume element dx changes from $d\xi$ to $rdrd\phi$, the new Hilbert space is

$$\left\{ u \in L^2(\mathbb{R}^+ \times \mathbb{T}, \mathbb{C}^2);\ \sqrt{r}u \in L^2(\mathbb{R}^+ \times \mathbb{T}, \mathbb{C}^2);\ u(r, \phi + 2\pi) = u(r, \phi) \right\}.$$

Half-angle rotation and conjugation by $\sqrt{r}$. Now put

$$w_\theta = \sqrt{r}\, R(-\tfrac{\phi}{2}) \hat{u}_\theta,$$

where

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

is the rotation by the angle ϕ. Using the relations

$$A(\phi) = R(\phi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R(-\phi), \qquad R'(\phi) = R(\phi + \tfrac{\pi}{2}),$$

we obtain the equivalent equation to (8) as

$$\left\{ e^{-2i\theta} r^2 - e^{i\theta} h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_r - e^{i\theta} \frac{h}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\phi - E \right\} w_\theta = 0. \tag{9}$$

The new Hilbert space is $\left\{ w \in L^2(\mathbb{R}^+ \times \mathbb{T}, \mathbb{C}^2);\ w(r, \phi + 2\pi) = -w(r, \phi) \right\}$.

Fourier series expansion. We expand w_θ in a Fourier series, taking into account that w_θ is 2π anti-periodic with respect to ϕ:

$$w_\theta(r,\phi) = \sum_{\tilde\nu \in \frac12 + \mathbb{Z}} e^{-i\tilde\nu\phi} w_{\theta,\nu}(r).$$

Then we see that the original eigenvalue problem (8) is reduced to that of the differential expression

$$P_{\theta,\nu} = e^{-2i\theta} r^2 + e^{i\theta} \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix}, \qquad \nu \in h(\mathbb{Z} + \tfrac12).$$

The resulting Hilbert space is $L^2(\mathbb{R}^+, \mathbb{C}^2)$.

Closedness of $P_{\theta,\nu}$ on $\tilde{\mathcal{D}}_\nu$. As before in the proof of Lemma 2.1 one shows a quadratic estimate, this time for the operator

$$T_{\alpha,\nu} := \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} + \alpha\, r^2$$

with $\alpha \in \mathbb{C}$, $\mathrm{Im}(\alpha) \neq 0$: there exist positive constants $b_{\alpha,\nu}, c_{\alpha,\nu} > 0$ such that for all $w \in C_c^\infty(\mathbb{R}^+, \mathbb{C}^2)$

$$\|T_{\alpha,\nu} w\|_{L^2}^2 + c_{\alpha,\nu}\|w\|_{L^2}^2 \geq \begin{cases} b_{\alpha,\nu}\left(\|hD_r w\|_{L^2}^2 + \|r^{-1}w\|_{L^2}^2 + \|r^2 w\|_{L^2}^2\right) & \text{if} \quad \nu \neq \pm\frac{h}{2}, \\ b_{\alpha,\nu}\left(\| \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} w\,\|_{L^2}^2 + \|r^2 w\|_{L^2}^2\right) & \text{if} \quad \nu = \pm\frac{h}{2}. \end{cases}$$

Indeed, for $\nu \in \{\pm\frac{3h}{2}, \pm\frac{5h}{2}, \dots\}$ one uses the lower bounds

$$\begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix}^2 = -h^2\Delta_r + \tfrac{\nu^2}{r^2} + ih\,\tfrac{\nu}{r^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \geq -h^2\Delta_r + \tfrac{\nu^2 - h|\nu|}{r^2},$$

$$\pm r^2 \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} \pm \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} r^2 \geq -|\alpha|^{-1}\left(-h^2\Delta_r + \tfrac{\nu^2 - h|\nu|}{r^2} + |\alpha|^2 r^4\right),$$

and

$$i\, r^2 \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} - i \begin{pmatrix} -hD_r & \frac{\nu}{r} \\ \frac{\nu}{r} & hD_r \end{pmatrix} r^2 = 2hr \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \geq -2rh.$$

Since $\nu^2 - h|\nu| > 0$ for $\nu \neq \pm\frac{h}{2}$, one obtains for suitable $b_{\alpha,\nu}, c_{\alpha,\nu} > 0$

$$\begin{aligned} T_{\bar\alpha,\nu} T_{\alpha,\nu} &\geq \left(1 - |\mathrm{Re}(\alpha)|\,|\alpha|^{-1}\right)\left(-h^2\Delta_r + \tfrac{\nu^2 - h|\nu|}{r^2} + |\alpha|^2 r^4\right) - |\mathrm{Im}(\alpha)| 2rh \\ &\geq b_{\alpha,\nu}\left(-h^2\Delta_r + r^{-2} + r^4\right) - c_{\alpha,\nu}. \end{aligned}$$

The estimate for $\nu = \pm\frac{h}{2}$ follows analogously. □

Proof of Proposition 1.1: Let E be a resonance of $P = -h^2\Delta + V$. By Lemma 2.3, there exist $\nu \in h(\mathbb{Z} + 1/2)$ and $w_\theta \in \tilde{\mathcal{D}}_\nu$ such that $P_{\theta,\nu} w_\theta = E w_\theta$. The origin $r = 0$ is a regular singular point for the operator $P_{\theta,\nu}$ with indicial roots $\pm\tilde\nu = \pm\nu/h$. Hence, by the theory of Fuchs, w_θ is a linear combination

$$w_\theta(r) = C_0 w^0(r) + C_\infty w^\infty(r)$$

of the two solutions $w^0(r)$ and $w^\infty(r)$ to the equation $P_{\theta,\nu} w = E w$ such that

$$w^0(r) \sim r^{|\tilde\nu|} \begin{pmatrix} 1 \\ -\mathrm{sgn}(\tilde\nu)\, i \end{pmatrix}, \quad w^\infty(r) \sim r^{-|\tilde\nu|} \begin{pmatrix} 1 \\ \mathrm{sgn}(\tilde\nu)\, i \end{pmatrix} \qquad (r \to 0),$$

see also §4. Then, the condition $w_\theta \in \tilde{\mathcal{D}_\nu}$ implies $C_\infty = 0$, and w_θ behaves like $r^{|\tilde{\nu}|}$ near the origin. In §5, the construction of complex WKB solutions with base points at infinity shows

$$w_\theta(r) = C_+ w_+^\infty(r) + C_- w_-^\infty(r)\,,$$

where $w_+^\infty(r)$ is exponentially growing and $w_-^\infty(r)$ exponentially decaying as $r \to \infty$. Hence, $C_+ = 0$, and $w_\theta(r)$ decays exponentially as $r \to \infty$. Setting $w(r) := w_\theta(e^{i\theta}r)$, one obtains a solution to $P_\nu w = Ew$ with the claimed properties.

Let $E \in \mathbb{C}$ and w be a solution of $P_\nu w = Ew$ for some $\nu \in h(\mathbb{Z}+1/2)$ such that $\lim_{r\to 0+} w(r) = 0$ and $r^2 w(e^{-i\theta}r), w'(e^{-i\theta}r) \in L^2(\mathbb{R}^+, \mathbb{C}^2)$. The preceding arguments yield, that $w_\theta(r) := w(e^{-i\theta}r)$ is a solution to the equation $P_{\theta,\nu} w_\theta = E w_\theta$ with $w_\theta \in \tilde{\mathcal{D}_\nu}$. By Lemma 2.3, E is a resonance of $P = -h^2\Delta + V$. □

Remark 2.4 Let $E \in \mathbb{C}$ and $\nu \in h(\mathbb{Z}+\frac{1}{2})$. Then the following equivalence holds: $u = (u_1, u_2)$ is a solution of $P_\nu u = Eu$, if and only if $\widetilde{u} = (-u_1, u_2)$ is a solution of $P_{-\nu}\widetilde{u} = E\,\widetilde{u}$. The same holds for the dilated operators $P_{\nu,\theta}$ and $P_{-\nu,\theta}$, $\theta \in]0, \frac{\pi}{3}[$. Hence, we will restrict our studies to the case $\nu \in h(\mathbb{N}-\frac{1}{2})$. ◇

3. Exact WKB method for 2 × 2 systems

We now wish to find a representation formula for the solutions of P_ν, from which it is possible to deduce the asymptotic expansion in h. The method is known as exact WKB method. We derive it in a somewhat more general context, and then apply it to our specific equation.

We study 2×2 systems of first order differential equations in a complex domain D, which are of the form

$$\begin{pmatrix} p_1(x) - \frac{h}{i}\frac{d}{dx} & \omega(x) \\ \omega(x) & p_2(x) + \frac{h}{i}\frac{d}{dx} \end{pmatrix} u(x) = 0\,, \tag{10}$$

or equivalently

$$\frac{h}{i}\frac{d}{dx} u(x) = \begin{pmatrix} p_1(x) & \omega(x) \\ -\omega(x) & -p_2(x) \end{pmatrix} u(x)\,. \tag{11}$$

The functions p_1, p_2, and ω are holomorphic in D. The following considerations will lead to the construction of exact WKB solutions for this type of systems.

3.1. **Formal construction.** A usual change of variables (using oscillatory part) and some other basic transformations reduce the operator to a more computable one. Let us put

$$g_+(x) = \tfrac{1}{2}(p_1(x) + p_2(x)) + \omega(x), \quad g_-(x) = -\tfrac{1}{2}(p_1(x) + p_2(x)) + \omega(x).$$

After conjugation by

$$M(x) = \exp\left(\frac{i}{2h}\int_0^x (p_1(t) - p_2(t))\,dt\right)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =: m(x)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

the system (10) is transformed into the trace-free system

$$\frac{h}{i}\frac{d}{dx} v(x) = \begin{pmatrix} 0 & g_+(x) \\ -g_-(x) & 0 \end{pmatrix} v(x)$$

with $u(x) = M(x)v(x)$. Introducing a new complex coordinate

$$z(x) = z(x;x_0) = \int_{x_0}^{x} \sqrt{g_+(t)g_-(t)}\, dt\,, \qquad x_0 \in D\,, \tag{12}$$

we look for solutions of the form $\mathrm{e}^{\pm\frac{z}{h}}\widetilde{w}_\pm(z)$.

Definition 3.1 Let p_1, p_2, and ω be holomorphic functions in D. The zeros of the function $g_+(x)g_-(x) = -\frac{1}{4}(p_1+p_2)^2+\omega^2$ are called the *turning points* of the system (10).

We note that due to the possible presence of such turning points the square root in the definition of $z(x)$ might be defined only *locally*. By formal calculations, the amplitude vector $\widetilde{w}_\pm(z)$ has to satisfy

$$\frac{h}{i}\frac{d}{dz}\widetilde{w}_\pm(z) = \begin{pmatrix} \pm i & H(z)^{-2} \\ -H(z)^2 & \pm i \end{pmatrix} \widetilde{w}_\pm(z)\,,$$

where the function H is given by

$$H(z(x)) = \left(\frac{g_-(x)}{g_+(x)}\right)^{1/4}\,.$$

For a decomposition with respect to image and kernel of the preceding system's matrix, we conjugate by

$$P_\pm(z) \;=\; 2^{-1}\begin{pmatrix} H(z) & \pm iH(z)^{-1} \\ H(z) & \mp iH(z)^{-1} \end{pmatrix}, \qquad P_\pm^{-1}(z) \;=\; \begin{pmatrix} H(z)^{-1} & H(z)^{-1} \\ \mp iH(z) & \pm iH(z) \end{pmatrix}$$

and obtain a system for $w_\pm(z) = P_\pm(z)\widetilde{w}_\pm(z)$,

$$\frac{d}{dz}w_\pm(z) = \begin{pmatrix} 0 & \frac{H'(z)}{H(z)} \\ \frac{H'(z)}{H(z)} & \mp\frac{2}{h} \end{pmatrix} w_\pm(z)\,,$$

where $H'(z)$ is shorthand for $dH(z)/dz$. The series ansatz

$$w_\pm(z) = \sum_{n\geq 0} \begin{pmatrix} w_{2n,\pm}(z) \\ w_{2n+1,\pm}(z) \end{pmatrix} \tag{13}$$

with $w_{0,\pm} \equiv 1$ and for $n \geq 1$, the recurrence equations

$$\left(\frac{d}{dz} \pm \frac{2}{h}\right) w_{2n+1,\pm}(z) = \frac{H'(z)}{H(z)}\, w_{2n,\pm}(z), \tag{14}$$

$$\frac{d}{dz}w_{2n+2,\pm}(z) = \frac{H'(z)}{H(z)}\, w_{2n+1,\pm}(z) \tag{15}$$

give us a formal solution up to some additive constants, which are fixed by setting

$$w_{n,\pm}(\widetilde{z}) = 0\,, \quad n \geq 1\,,$$

for a base point $\widetilde{z} = z(\widetilde{x})$ where $\widetilde{x} \in D$ is not a turning point. We note that the preceding equations for $w_{n,\pm}$ are the same as the ones obtained by an exact WKB construction for scalar Schrödinger equations. See for example the work of C. Gérard and A. Grigis [15] or T. Ramond [29].

Let Ω be a simply connected subset of D which does not contain any turning point. Then the function $z = z(x)$ is conformal from Ω onto $z(\Omega)$. Assume that $\tilde{z} \in z(\Omega)$. If

$\Gamma_\pm(\widetilde{z}, z)$ denotes a path of finite length in $z(\Omega)$ connecting $\widetilde{z}$ and $z \in z(\Omega)$, we can formally rewrite the above differential equations for $n \geq 0$ as

$$\begin{aligned} w_{2n+1,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \exp\left(\pm\tfrac{2}{h}(\zeta - z)\right) \frac{H'(\zeta)}{H(\zeta)}\, w_{2n,\pm}(\zeta)\, d\zeta, \\ w_{2n+2,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \frac{H'(\zeta)}{H(\zeta)}\, w_{2n+1,\pm}(\zeta)\, d\zeta \end{aligned}$$

or after iterated integration as

$$\begin{aligned} w_{2n+1,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \int_{\Gamma_\pm(\widetilde{z},\zeta_{2n+1})} \dots \int_{\Gamma_\pm(\widetilde{z},\zeta_1)} \exp\left(\pm\tfrac{2}{h}(\zeta_2 - \zeta_3 + \ldots + \zeta_{2n+1} - z)\right) \times \\ &\qquad \times \frac{H'(\zeta_1)}{H(\zeta_1)} \cdots \frac{H'(\zeta_{2n+1})}{H(\zeta_{2n+1})}\, d\zeta_1 \ldots d\zeta_{2n+1}, \\ w_{2n+2,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \int_{\Gamma_\pm(\widetilde{z},\zeta_{2n+2})} \dots \int_{\Gamma_\pm(\widetilde{z},\zeta_1)} \exp\left(\pm\tfrac{2}{h}(\zeta_2 - \zeta_3 + \ldots - \zeta_{2n+2})\right) \times \\ &\qquad \times \frac{H'(\zeta_1)}{H(\zeta_1)} \cdots \frac{H'(\zeta_{2n+2})}{H(\zeta_{2n+2})}\, d\zeta_1 \ldots d\zeta_{2n+2}\,. \end{aligned}$$

3.2. **Convergence, h-dependence, and Wronskians.** We now give the preceding formal construction some mathematical meaning in turning point-free compact sets $\Omega \subset D$.

Lemma 3.2 *For any fixed $h > 0$, the formal series (13) converges uniformly in any compact subset of Ω, and*

$$w_\pm^{\text{even}}(x,h) = \sum_{n\geq 0} w_{2n,\pm}(z(x)), \quad w_\pm^{\text{odd}}(x,h) = \sum_{n\geq 0} w_{2n+1,\pm}(z(x)) \tag{16}$$

are holomorphic functions in D.

Proof: In Ω, all the functions defined above are well-defined analytic functions. For compact subsets $K \subset \Omega$ and $\widetilde{z}, z \in z(K)$ there exist positive constants $C_\pm^h(K) > 0$ depending on the semiclassical parameter h and the compact K such that

$$\sup_{\zeta \in \Gamma_\pm(\widetilde{z},z)} \left| \exp\left(\pm\tfrac{2}{h}\zeta\right) \frac{H'(\zeta)}{H(\zeta)} \right| \leq C_\pm^h(K)\,. \tag{17}$$

If we denote the maximal length of the paths $\Gamma_\pm(\widetilde{z}, \cdot) \subset K$ in the preceding iterated integrations by $0 < L < \infty$, then

$$\sup_{z \in z(K)} |w_{n,\pm}(z)| \leq \frac{C_\pm^h(K)^n L^n}{n!}, \qquad n \geq 0\,,$$

where the bound $\frac{L^n}{n!}$ comes from the volume of a simplex with length L. □

Thus, we have uniform convergence of the series (13) for $w_\pm(z)$ and exact solutions

$$u_\pm(x) = \mathrm{e}^{\pm\frac{z(x)}{h}}\, m(x)\, T_\pm(z(x)) \begin{pmatrix} w_\pm^{\text{even}}(x) \\ w_\pm^{\text{odd}}(x) \end{pmatrix}$$

of the original problem (10) on turning point free sets Ω, where

$$
\begin{aligned}
T_\pm(z) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} H(z)^{-1} & H(z)^{-1} \\ \mp iH(z) & \pm iH(z) \end{pmatrix} \\
&= \begin{pmatrix} H(z)^{-1} \mp iH(z) & H(z)^{-1} \pm iH(z) \\ -H(z)^{-1} \mp iH(z) & -H(z)^{-1} \pm iH(z) \end{pmatrix}, \qquad z \in z(\Omega)\,.
\end{aligned} \tag{18}
$$

We write these solutions $u_\pm(x)$ as

$$u_\pm(x; x_0, \widetilde{x})$$

indicating the particular choice of the phase base point x_0 in (12), which defines the phase function $z(x) = z(x; x_0)$, and the choice of the amplitude base point $\widetilde{z} = z(\widetilde{x})$, which is the initial point of the path $\Gamma_\pm(\widetilde{z}, \cdot)$.

For a fixed $\widetilde{x} \in \Omega$, let $\Omega_\pm$ be the set of $x \in \Omega$ such that there exists a path $\Gamma_\pm(z(\widetilde{x}), z(x))$ along which $x \mapsto \pm\mathrm{Re}\, z(x)$ increases strictly. Then,

Proposition 3.3 *The identities (16) for $w_\pm^{\mathrm{even}}(x,h)$ and $w_\pm^{\mathrm{odd}}(x,h)$ give asymptotic expansions in $\Omega_\pm$. More precisely, we have for any $\alpha \in \mathbb{N}$ and $N \in \mathbb{N}$*

$$\partial^\alpha \big(w_\pm^{\mathrm{even}}(x,h) - \sum_{n=0}^{N} w_{2n,\pm}(z(x)) \big) = O(h^{N+1}),$$

$$\partial^\alpha \big(w_\pm^{\mathrm{odd}}(x,h) - \sum_{n=0}^{N} w_{2n+1,\pm}(z(x)) \big) = O(h^{N+2}).$$

uniformly in compact subsets of $\Omega_\pm$. In particular,

$$w_\pm^{\mathrm{even}}(x,h) = 1 + O(h), \quad w_\pm^{\mathrm{odd}}(x,h) = O(h).$$

The proof is just the same as that of Proposition 1.2 of [15]. The key point is the following: Since the iterated integrations defining $w_{n,\pm}(z)$ contain terms of the form $\exp(\pm\zeta/h)$, one has to make sure that $\zeta \mapsto \pm\mathrm{Re}(\zeta)$ is a strictly increasing function along the path $\Gamma_\pm(\widetilde{z}, z)$. In other words, the paths $\Gamma_\pm(z(\widetilde{x}), z(x))$ have to intersect the *Stokes lines*, that is the level curves of $x \mapsto \mathrm{Re}(z(x))$, transversally in a suitable direction.

One defines the Wronskian of two $\mathbb{C}^2$-valued functions u, v as $\mathcal{W}(u,v) = u_1 v_2 - u_2 v_1$. If $z = \alpha u + \beta v$ with $\alpha, \beta \in \mathbb{C}$, then

$$\alpha = \frac{\mathcal{W}(z,v)}{\mathcal{W}(u,v)}\,, \qquad \beta = -\frac{\mathcal{W}(z,u)}{\mathcal{W}(u,v)}\,.$$

Elementary computations give the following exact Wronskian formulas for exact WKB solutions with different phase and amplitude base points, using $w_\pm^{\mathrm{even}}$ and $w_\pm^{\mathrm{odd}}$.

Lemma 3.4 *Let $x, x_0, y_0, \widetilde{x}, \widetilde{y} \in \Omega$. Then,*

$$
\begin{aligned}
\mathcal{W}(u_\pm(x; x_0, \widetilde{x}), u_\pm(x; y_0, \widetilde{y})) \;&=\; \pm 2\, i\, m(x)^2 \;\exp\left(\pm\tfrac{1}{h}\,(z(x;x_0) + z(x;y_0))\right) \\
&\times \Big(w_\pm^{\mathrm{even}}(x; x_0, \widetilde{x}) w_\pm^{\mathrm{odd}}(x; y_0, \widetilde{y}) - w_\pm^{\mathrm{odd}}(x; x_0, \widetilde{x}) w_\pm^{\mathrm{even}}(x; y_0, \widetilde{y}) \Big)\;,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\mathcal{W}(u_\pm(x; x_0, \widetilde{x}), u_\mp(x; y_0, \widetilde{y})) \;&=\; \pm 2\, i\, m(x)^2 \;\exp\left(\pm\tfrac{1}{h}\,(z(x;x_0) - z(x;y_0))\right) \\
&\times \Big(w_\pm^{\mathrm{even}}(x; x_0, \widetilde{x}) w_\mp^{\mathrm{even}}(x; y_0, \widetilde{y}) - w_\pm^{\mathrm{odd}}(x; x_0, \widetilde{x}) w_\mp^{\mathrm{odd}}(x; y_0, \widetilde{y}) \Big)\;.
\end{aligned} \tag{20}
$$

In particular, if $p_1 = p_2$, then $m = 1$, all the Wronskians are independent of x, and we have for solutions with the same phase base point

$$\mathcal{W}(u_\pm(\cdot\,;x_0,\widetilde{x}), u_\pm(\cdot\,;x_0,\widetilde{y})) \;=\; \mp 2\,i\,\exp\left(\pm\tfrac{2}{h}z(\widetilde{y};x_0)\right) w_\pm^{\mathrm{odd}}(\widetilde{y}\,;x_0,\widetilde{x})\,, \tag{21}$$

$$\mathcal{W}(u_\pm(\cdot\,;x_0,\widetilde{x}), u_\mp(\cdot\,;x_0,\widetilde{y})) \;=\; \pm 2\,i\; w_\pm^{\mathrm{even}}(\widetilde{y}\,;x_0,\widetilde{x})\,. \tag{22}$$

4. Formulation of the resonance condition

We now go back to our system

$$(P_\nu - E)\,u(r) \;=\; \begin{pmatrix} r^2 - E - \frac{h}{i}\frac{d}{dr} & \frac{\nu}{r} \\ \frac{\nu}{r} & r^2 - E + \frac{h}{i}\frac{d}{dr} \end{pmatrix} u(r) \;=\; 0\,, \qquad r\in\mathbb{R}^+\,, \tag{23}$$

for $E\in\mathbb{C}$ and $\nu\in h(\mathbb{N}-\frac{1}{2})$, and study in particular their behaviour as $r\to 0$ and $r\to\infty$.

In what follows, we use x instead of r and rewrite the equation (23) in the form

$$hD_x u = Au, \quad A = \begin{pmatrix} x^2-E & \nu/x \\ -\nu/x & E-x^2 \end{pmatrix}. \tag{24}$$

The origin is a regular singular point of the equation with indicial roots $\pm\tilde{\nu}$, where $\tilde{\nu} := \nu/h > 0$ is positive, see Remark 2.4. Indeed, (24) can be rewritten as

$$x\frac{d}{dx}u = (A_0 + O(x))u \quad (x\to 0), \quad A_0 = \begin{pmatrix} 0 & i\tilde{\nu} \\ -i\tilde{\nu} & 0 \end{pmatrix},$$

and $\pm\tilde{\nu}$ are the eigenvalues of A_0, and ${}^t(1,\mp i)$ are the corresponding eigenvectors. Let $u_0(x)$ be a solution corresponding to the index $\tilde{\nu}$,

$$u_0(x)\sim x^{\tilde{\nu}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

as $x\to 0$. We will see in the next section, that there exist Jost solutions $f^\pm(x)$, which are characterised by their asymptotic behaviour at infinity,

$$f^+(x)\;\sim\; e^{+i\,(x^3-3Ex)/3h}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f^-(x)\;\sim\; e^{-i\,(x^3-3Ex)/3h}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

as $x\to+\infty$. If $\theta\in]0,\pi/3[$, then $f^+(xe^{-i\theta})$ is exponentially growing and $f^-(xe^{-i\theta})$ exponentially decaying as $x\to+\infty$. Since f^+ and f^- are linearly independent, u_0 can be expressed as a linear combination of these solutions,

$$u_0(x) = c^+(E,h)f^+(x) + c^-(E,h)f^-(x). \tag{26}$$

¿From Definition 2.2 and Proposition 1.1, we obtain the following characterization of resonances:

Proposition 4.1 *The energy $E\in\mathbb{C}$ is a resonance of P if and only if there exists $\nu\in h(\mathbb{N}-\frac{1}{2})$ with $c^+(E,h)=0$.*

To calculate the coefficients $c^\pm(E,h)$, which connect the solution u_0 defined at the origin with the Jost solutions $f^\pm$ defined at infinity, we need some intermediate solutions, which

we will construct as exact WKB solutions. Let us recall the WKB construction of §3 in this case. Note that $\operatorname{tr} A = 0$ and so $m = 1$. Exact WKB solutions are of the form

$$u_\pm(x; x_0, \tilde{x}) = e^{\pm z(x)/h} T_\pm(z(x)) \begin{pmatrix} w_\pm^{\mathrm{even}}(x) \\ w_\pm^{\mathrm{odd}}(x) \end{pmatrix}, \tag{27}$$

where the phase function $z(x)$ is defined by

$$z(x) = z(x; x_0) = \int_{x_0}^{x} \sqrt{g_+(t) g_-(t)}\, dt, \qquad g_\pm(x) = \frac{\nu}{x} \mp E \pm x^2$$

for a phase base point x_0, $T_\pm(z(x))$ is a 2×2 matrix defined by (18) with

$$H(z(x)) = \left(\frac{g_-(x)}{g_+(x)} \right)^{1/4} = \left(\frac{\nu + Ex - x^3}{\nu - Ex + x^3} \right)^{1/4},$$

and the symbols $w_\pm^{\mathrm{even}}(x, h) = \sum_{n \geq 0} w_{2n,\pm}(z(x))$ and $w_\pm^{\mathrm{odd}}(x, h) = \sum_{n \geq 0} w_{2n+1,\pm}(z(x))$ are constructed by recursive integrations with an amplitude base point $\tilde{x}$.

There are at most six turning points, the zeros of $x \mapsto g_+(x) g_-(x)$, in the whole complex plane $\mathbb{C}_x$. They are point-symmetric with respect to the origin and denoted by $\{\pm r_j\}_{j=0}^{2}$, see Appendix A. For $E > 0$ fixed and $\nu > 0$ sufficiently small, they are real and satisfy

$$0 < r_0 < r_1 < \sqrt{E} < r_2.$$

As $h \to 0$, that is $\nu \to 0$, r_0 tends to 0 and r_1 and r_2 tend to $\sqrt{E}$, where $\sqrt{E}$ denotes the square root of $E \in \mathbb{C}$ in the right half-plane. Notice that r_0 and r_1 are zeros of $g_+(x)$ and r_2 is a zero of $g_-(x)$.

The Stokes curves are the level curves of the function $x \mapsto \operatorname{Re} z(x)$. The Stokes curves emanating from the turning points are drawn in Figure 2.

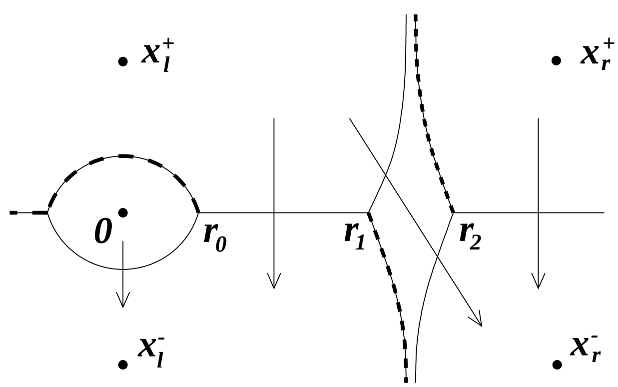

FIGURE 2. The Stokes curves, that is the level curves of $x \mapsto \operatorname{Re} z(x)$, which emanate from the turning points r_0, r_1, r_2. The arrows indicate the directions along which $\operatorname{Re} z(x)$ increases. The dashed lines show the branch cuts. $x_l^\pm$ and $x_r^\pm$ are the amplitude base points for the constructed exact WKB solutions.

We put branch cuts as drawn in Figure 2 for the multi-valued functions

$$\sqrt{g_+(x) g_-(x)} = \frac{\sqrt{\nu^2 - x^2(E - x^2)^2}}{x}, \qquad H(z(x)) = \left(\frac{\nu + Ex - x^3}{\nu - Ex + x^3} \right)^{1/4},$$

and suppose that

$$\sqrt{\nu^2 - x^2(E-x^2)^2}|_{x=0} = \nu, \qquad \left(\frac{\nu + Ex - x^3}{\nu - Ex + x^3}\right)^{1/4}|_{x=0} = 1.$$

Let $E > 0$ be positive, and let $\nu = \tilde{\nu}h > 0$ be sufficiently small. Then, for example, $\sqrt{g_+(x)g_-(x)} \in i\,\mathbb{R}^+$ for $r_0 < x < r_1$ and $r_2 < x$, and $H(z(x)) \in e^{-i\pi/4}\,\mathbb{R}^+$ for $r_0 < x < r_1$, while $H(z(x)) \in e^{i\pi/4}\mathbb{R}^+$ for $r_2 < x$.

Choosing amplitude base points $x_l^\pm$, $x_r^\pm$ as in Figure 2 ($x_l^\pm$ are supposed to be purely imaginary for a technical reason in the proof of Lemma 6), we define the following six exact WKB solutions

$$\begin{aligned} u_0^\pm(x) &= u_\pm(x; r_0, x_l^\pm), \\ u_l^\pm(x) &= u_\pm(x; r_1, x_l^\pm), \\ u_r^\pm(x) &= u_\pm(x; r_2, x_r^\pm) \end{aligned}$$

with the turning points r_0, r_1, r_2 as phase base points. The three pairs $(u_0^+(x), u_0^-(x))$, $(u_l^+(x), u_l^-(x))$, $(u_r^+(x), u_r^-(x))$ are all linearly independent and they are connected with u_0 and (f^+, f^-) by transfer matrices ${}^t(c_0^+(E,h), c_0^-(E,h))$, $T_1(E,h)$, $T_2(E,h)$, $T_3(E,h)$:

$$u_0(x) = (u_0^+(x), u_0^-(x)) \begin{pmatrix} c_0^+(E,h) \\ c_0^-(E,h) \end{pmatrix}, \tag{28}$$

$$(u_0^+(x), u_0^-(x)) = (u_l^+(x), u_l^-(x)) T_1(E,h) \tag{29}$$

$$(u_l^+(x), u_l^-(x)) = (u_r^+(x), u_r^-(x)) T_2(E,h) \tag{30}$$

$$(u_r^+(x), u_r^-(x)) = (f^+(x), f^-(x)) T_3(E,h) \tag{31}$$

Then, the coefficients $c^+(E,h)$, $c^-(E,h)$ in (26) are given by

$$\begin{pmatrix} c^+ \\ c^- \end{pmatrix} = T_3 T_2 T_1 \begin{pmatrix} c_0^+ \\ c_0^- \end{pmatrix}. \tag{32}$$

Now we can describe the strategy for the remainder of this article: The matrix T_3 for the transfer at infinity is computed in Section 5, the connection coefficients $c_0^\pm$ at the origin in Section 6, and the transfer matrix T_2 near $x = \sqrt{E}$ in Section 7. The matrix T_1 is the easiest one, and can be determined right away using Lemma 3.4. Because $u_0^\pm$ and $u_l^\pm$ differ only in the base point of the phase, one uses (19) and (20) for $x = x_l^\pm$ to obtain

$$T_1 = \begin{pmatrix} e^{S_{01}/h} & 0 \\ 0 & e^{-S_{01}/h} \end{pmatrix}, \tag{33}$$

where

$$S_{01}(E,h) = \int_{r_0}^{r_1} \sqrt{g_+(x)g_-(x)}dx = \int_{r_0}^{r_1} \frac{\sqrt{\nu^2 - x^2(E-x^2)^2}}{x}dx$$

is the action integral between the turning points r_0 and r_1.

5. Jost solutions

The Jost solutions of $P_\nu u = Eu$ are characterised by their behaviour (25) at infinity. They can be expressed as exact WKB solutions with the base points of both phase and amplitude placed at infinity. This fact allows us to calculate T_3.

First we define the phase function with base point at infinity,

$$z(x,\infty) = \int_{+\infty}^{x} \left(\sqrt{\nu^2 - t^2(E-t^2)^2}/t - i(t^2 - E) \right) dt + \tfrac{i}{3}(x^3 - 3Ex).$$

Taking the branch of the square root into account, we see that the integral converges absolutely, hence

$$z(x,\infty) = \frac{i}{3}(x^3 - 3Ex) + o(1) \qquad (x \to +\infty).$$

Next we define amplitudes based at infinity. The Stokes curves are asymptotically like horizontal lines $\{\mathrm{Im} x = \text{const.}\}$, and $\mathrm{Im} x \mapsto \mathrm{Re} z(x)$ is a decreasing function (see Figure 2). As in Section 3 of [29], we choose infinite paths $\gamma_\pm(x)$ starting from infinity and ending at x, which are asymptotically like lines of the form $\{\mathrm{Im} x = \mp\delta\, \mathrm{Re} x\}$ for some $\delta > 0$, such that $x \mapsto \mp \mathrm{Re} z(x)$ are strictly increasing functions along $\gamma_\pm(x)$. Denoting the path $z(\gamma_\pm(x))$ by $\Gamma_\pm(+\infty, z(x))$ and setting $w_{0,\pm} \equiv 1$, we inductively define $w_{n,\pm}(z)$ by

$$\begin{aligned} w_{2n+1,\pm}(z) &= \int_{\Gamma_\pm(+\infty,z)} \exp\left(\pm\tfrac{2}{h}(\zeta - z)\right) \frac{H'(\zeta)}{H(\zeta)}\, w_{2n,\pm}(\zeta)\, d\zeta, \\ w_{2n+2,\pm}(z) &= \int_{\Gamma_\pm(+\infty,z)} \frac{H'(\zeta)}{H(\zeta)}\, w_{2n+1,\pm}(\zeta)\, d\zeta\,, \qquad n \geq 0\,. \end{aligned}$$

Noticing that

$$\frac{H'(x)}{H(x)} = \frac{\nu}{2}\, \frac{E - 3x^2}{(\nu - Ex + x^3)^2} = O(x^{-4})$$

as $x \to +\infty$, one constructs well-defined exact WKB solutions $u_\infty^\pm(x)$ corresponding to these base points, proceeding as in Section 3 of [29]. Up to a constant prefactor, $u_\infty^\pm(x)$ are the previously defined Jost solutions:

Lemma 5.1 *Let $u_\infty^\pm$ be the exact WKB solutions with phase and amplitude base point at infinity, $f^\pm$ the Jost solutions defined in (25). Then,*

$$f^\pm(x) = \pm\tfrac{1}{2}\, e^{\pi i/4}\, u_\infty^\pm(x)\,, \qquad x > 0\,.$$

Proof: We just check the asymptotic behavior of $u_\infty^\pm$ at infinity. Since $H(z(x)) \to e^{\pi i/4}$ as $x \to +\infty$, we get by an elementary calculation

$$u_\infty^\pm(x) \;\sim\; e^{\pm i(x^3 - 3Ex)/3h} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-\pi i/4} & e^{-\pi i/4} \\ \mp i e^{\pi i/4} & \pm i e^{\pi i/4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

that is,

$$u_\infty^+(x) \sim 2e^{-\pi i/4} e^{+i(x^3-3Ex)/3h} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_\infty^-(x) \sim -2e^{-\pi i/4} e^{-i(x^3-3Ex)/3h} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $x \to +\infty$. □

As an immediate consequence of the behaviour of the solutions $u_\infty^\pm(x)$ for $x \to +\infty$, we obtain that the discrete spectrum of the full operator P is empty.

Proposition 5.2 *Let P be the full operator as defined in (6). Then, $\sigma_{\rm disc}(P) = \emptyset$.*

Proof: The proof of Lemma 2.3 shows, that $E \in \mathbb{R}$ is an eigenvalue of the self-adjoint operator P if and only if there exists $\nu \in h(\mathbb{N} - \frac{1}{2})$ such that $E \in \mathbb{R}$ is an eigenvalue of the ordinary differential operator P_ν. At infinity, every distributional solution of (23) has to be a linear combination of $f^\pm$. However, for $E \in \mathbb{R}$ neither f^+ nor f^- are in $L^2(\mathbb{R}^+, \mathbb{C}^2)$. Hence, the discrete spectrum is empty. □

The main result of this section is the following proposition.

Proposition 5.3 *There exists a positive $\delta > 0$ independent from $E \in \mathbb{C}$ and $h > 0$, such that the transfer matrix T_3 defined in (31) satisfies*

$$T_3(E,h) = 2e^{-\pi i/4} \begin{pmatrix} e^{S_{2\infty}(E,h)/h}\,(1 + O(h)) & O(e^{-\delta/h}) \\ O(e^{-\delta/h}) & e^{-S_{2\infty}(E,h)/h}\,(1+O(h)) \end{pmatrix}$$

as $h \to 0$, where $S_{2\infty}(E,h)$ is the action between r_2 and $+\infty$,

$$S_{2\infty}(E,h) = \int_{r_2}^{+\infty} \left(\sqrt{\nu^2 - x^2(E-x^2)^2}/x - i(x^2 - E) \right) dx + \tfrac{i}{3}(r_2^3 - 3Er_2).$$

Proof: We use the Wronskian formulas of Lemma 3.4 and the asymptotic expansions of Proposition 3.3 to calculate T_3. Setting

$$(u_r^+, u_r^-) =: (u_\infty^+, u_\infty^-)\,\tilde{T}_3\,,$$

the previous Lemma 5.1 gives

$$T_3 = \begin{pmatrix} 2e^{-\pi i/4} & 0 \\ 0 & -2e^{-\pi i/4} \end{pmatrix} \tilde{T}_3\,.$$

The difference of the phases $z(x; r_2)$ and $z(x; +\infty)$ is the action $S_{2\infty}(E,h)$, and we have

$$\mathcal{W}(u_r^\pm, u_\infty^\mp) = \pm 2i\, e^{\pm S_{2,\infty}/h}\,(1 + O(h))\,, \qquad \mathcal{W}(u_\infty^\pm, u_\infty^\mp) = \pm 2i\,(1+O(h))\,.$$

Since there exists $\delta > 0$ with

$$\mathcal{W}(u_r^\pm, u_\infty^\pm) = O(e^{-\delta/h}),$$

one obtains

$$\tilde{T}_3 = \begin{pmatrix} e^{S_{2\infty}/h}\,(1+O(h)) & O(e^{-\delta/h}) \\ O(e^{-\delta/h}) & -e^{-S_{2\infty}/h}\,(1+O(h)) \end{pmatrix}.$$

□

6. Asymptotics of the subdominant solution near the origin

We recall that the origin is a regular singular point of P_ν with Fuchs indices $\pm\tilde{\nu}$, and that u_0 is a solution corresponding to $\tilde{\nu}$, which is unique up to a constant multiple. The purpose of this section is to calculate the connection coefficients $c_0^{\pm}(E,h)$ in (28), i.e. to connect u_0 with the exact WKB solutions $u_0^+(x)$ and $u_0^-(x)$. For this, we first need to express u_0 as an exact WKB solution.

Let us look at the asymptotic behaviour of the phase $z(x)=z(x;r_0)$ as $x\to 0$ for a fixed $h>0$. Since $\sqrt{\nu^2-x^2(E-x^2)^2}=\nu+x\phi(x)$ with $\phi(x)$ holomorphic near $x=0$,

$$e^{z(x)/h}=\exp\left(\int_{r_0}^{x}(\nu/t+\phi(t))\,dt/h\right)=C_{E,h}\,x^{\tilde{\nu}}\exp\left(\int_0^x\phi(t)dt/h\right)$$

where $C_{E,h}:=r_0^{-\tilde{\nu}}\exp(-\int_0^{r_0}\phi(t)dt/h)>0$ is a positive constant. Moreover $H(x)$ is a holomorphic function at $x=0$ with $H(0)=1$. Thus,

$$e^{z(x)/h}\,T_+(z(x))\begin{pmatrix}1\\0\end{pmatrix}\sim C_{E,h}\,x^{\tilde{\nu}}\begin{pmatrix}1\\-i\end{pmatrix},\qquad x\to 0,\tag{34}$$

while $h>0$ fixed. This suggests that u_0 is collinear to an exact WKB solution of the type $+$. However, it is important to notice that the amplitude base point $\tilde{x}$ of the WKB solution should be placed at the origin in order to have

$$\begin{pmatrix}w_+^{\mathrm{even}}(x)\\w_+^{\mathrm{odd}}(x)\end{pmatrix}\sim\begin{pmatrix}1\\0\end{pmatrix},\qquad x\to 0,$$

since $x\mapsto\operatorname{Re}z(x)$ is decreasing as x tends to 0 in radial directions (see Figure 2). Moreover, the origin is a singular point for the equation and we need to check that the exact WKB solution $u_+(x;r_0,0)$ is well defined, i.e. the recurrence equations (14), (15) with initial value $w_{0,+}^0\equiv 1$, $w_{n,+}^0(0)=0$, $n\geq 1$, define a sequence of holomorphic functions $\{w_{n,+}^0\}_{n\geq 0}$, and the series $\sum_{n\geq 0}w_{2n,+}^0(x)$ and $\sum_{n\geq 0}w_{2n+1,+}^0(x)$ converge in a neighborhood of the origin. We rewrite (14), (15) with respect to x:

$$\left(\frac{d}{dx}+\frac{2}{h}\sqrt{g_+(x)g_-(x)}\right)w_{2n+1,+}^0(x)=\frac{H_x'(x)}{H(x)}\,w_{2n,+}^0(x),\tag{35}$$

$$\frac{d}{dx}w_{2n+2,+}^0(x)=\frac{H_x'(x)}{H(x)}\,w_{2n+1,+}^0(x),\tag{36}$$

where H_x' stands for the derivative of H with respect to x.

These equations are of the form

$$\frac{dw}{dx}+\frac{b(x)}{x}w=f(x),\quad w(0)=0$$

with $b(x)$ and $f(x)$ given holomorphic functions at the origin. In our case, $b\equiv 0$ for (36) or $b(x)=2x\sqrt{g_+(x)g_-(x)}/h$, $b(0)=2\tilde{\nu}$ for (35). This Cauchy problem has a unique holomorphic solution if $\operatorname{Re}b(0)>-1$, and the solution is given by

$$w(x)=x\int_0^1 t^{b(0)}\exp\left(-x\int_t^1\tilde{b}(xs)ds\right)f(xt)\,dt,\tag{37}$$

where $\tilde{b}(x)=(b(x)-b(0))/x$.

Hence for $\tilde{\nu}>0$, $\{w_{n,+}^0\}$ are uniquely determined and given by the recursive integrals

$$w_{2n+2,+}^0=I_0(w_{2n+1,+}^0),\quad w_{2n+1,+}^0=I_1(w_{2n,+}^0),$$

where

$$I_0(f) = \int_0^x \frac{H'_\xi(\xi)}{H(\xi)} f(\xi) d\xi = \frac{1}{2} \int_0^x \frac{\nu(E-3\xi^2)}{\nu^2 - \xi^2(E-\xi^2)^2} f(\xi) d\xi,$$

$$I_1(f) = \int_0^x \exp\left(-\frac{2}{h}\int_\xi^x \sqrt{g_+(t)g_-(t)}dt\right) \frac{H'_\xi(\xi)}{H(\xi)} f(\xi) d\xi$$

$$= \frac{1}{2}\int_0^x e^{-2\int_\xi^x \sqrt{\nu^2/t^2-(E-t^2)^2}dt/h} \frac{\nu(E-3\xi^2)}{\nu^2-\xi^2(E-\xi^2)^2} f(\xi)d\xi. \tag{38}$$

It is not difficult to see that for any fixed positive $h > 0$, the series

$$w_{+,0}^{\mathrm{even}}(x,h) = \sum_{n=0}^\infty w_{2n,+}^0(x), \quad w_{+,0}^{\mathrm{odd}}(x,h) = \sum_{n=0}^\infty w_{2n+1,+}^0(x).$$

are absolutely convergent in a sufficiently small neighborhood of the origin. Hence, the function

$$\tilde{u}_0(x) := e^{z(x)/h} T_+(z(x)) \begin{pmatrix} w_{+,0}^{\mathrm{even}}(x) \\ w_{+,0}^{\mathrm{odd}}(x) \end{pmatrix}$$

defines a solution to (24).

Next, we study the asymptotic behaviour of the connection coefficients $c_0^\pm(E,h)$ as $h \to 0$, using the exact WKB solution $\tilde{u}_0$. For that purpose we will need some bounds on $w_{+,0}^{\mathrm{even}}(x,h)$ and $w_{+,0}^{\mathrm{odd}}(x,h)$. Since we will finally use those in the Wronskian formulas, it is enough to deal with the case when x is purely imaginary, $x = iR$ with $R > 0$. We start by the following elementary estimate.

Lemma 6.1 *For any $\kappa \geq 0$, $m > 0$ and $\tau > 0$, one has*

$$\int_0^\tau e^{\kappa(r-\tau)} \frac{r^{m-1}}{(1+r)^{m+1}} dr \leq \frac{1}{m}\left(\frac{\tau}{1+\tau}\right)^m.$$

Proof: One integrates by parts,

$$\begin{aligned}\int_0^\tau e^{\kappa(r-\tau)} \frac{r^{m-1}}{(1+r)^{m+1}} dr &= \frac{\tau^m}{m(1+\tau)^{m+1}} - \frac{1}{m}\int_0^\tau r^m \frac{d}{dr}\left(\frac{e^{\kappa(r-\tau)}}{(1+r)^{m+1}}\right) dr \\ &\leq \frac{1}{m}\left(\frac{\tau}{1+\tau}\right)^m,\end{aligned}$$

using that the subtracted integral is positive. □

Lemma 6.2 *For any $E > 0$, $\tau > 0$, and $n \in \mathbb{N}$, one has*

$$|w_{n,+}^0(i\nu\tau/E)| \leq \frac{K(\tau)^n}{n!}\left(\frac{\tau}{1+\tau}\right)^n, \tag{39}$$

where $K(\tau) = 1 + 3\nu^2\tau^2/E^3$.

Proof: For $x = iR$, $R > 0$, we have by the changes of variables $t = is$, $\xi = i\rho$,

$$I_0(f)|_{x=iR} = \frac{1}{2i}\int_0^R \frac{\nu(E+3\rho^2)}{\nu^2+\rho^2(E+\rho^2)^2} f(i\rho) d\rho,$$

$$I_1(f)|_{x=iR} = \frac{1}{2i}\int_0^R \exp\left\{-\frac{2}{h}\int_\rho^R \frac{\sqrt{\nu^2+s^2(E+s^2)^2}}{s}\,ds\right\}\frac{\nu(E+3\rho^2)}{\nu^2+\rho^2(E+\rho^2)^2}d\rho.$$

Since

$$\nu^2+s^2(E+s^2)^2 > E^2s^2, \quad \frac{\nu(E+3\rho^2)}{\nu^2+\rho^2(E+\rho^2)^2} \le \frac{\nu(E+3R^2)}{\nu^2+\rho^2E^2},$$

we have for $R=\nu\tau/E$,

$$|I_0(f)|_{x=i\nu\tau/E}| \le \frac{K(\tau)}{2}\int_0^\tau \frac{1}{1+r^2}\,|f\,(i\nu r/E)|\,dr, \tag{40}$$

$$|I_1(f)|_{x=i\nu\tau/E}| \le \frac{K(\tau)}{2}\int_0^\tau e^{2\tilde\nu(r-\tau)}\frac{1}{1+r^2}\,|f\,(i\nu r/E)|\,dr. \tag{41}$$

We now proceed by induction over $n\in\mathbb{N}$. Since $w^0_{0,+}\equiv 1$, inequality (39) is trivially satisfied for $n=0$. Next, we assume that (39) holds for $n=2k$. Then, from (41) and Lemma 6.1, one has

$$\begin{aligned}|w^0_{2k+1,+}(i\nu\tau/E)| &= |I_1(w^0_{2k,+})|_{x=i\nu\tau/E}| \le \frac{K(\tau)}{2}\int_0^\tau e^{2\tilde\nu(r-\tau)}\frac{1}{1+r^2}|w^0_{2k,+}(i\nu r/E)|dr\\ &\le \frac{K(\tau)^{2k+1}}{2k!}\int_0^\tau e^{2\tilde\nu(r-\tau)}\frac{r^{2k}}{(1+r)^{2k+2}}dr \le \frac{K(\tau)^{2k+1}}{(2k+1)!}\left(\frac{\tau}{1+\tau}\right)^{2k+1}.\end{aligned}$$

Thus (39) holds for $n=2k+1$. In the same way, we can show that it holds for $n=2k+2$. □

Proposition 6.3 *There exists a non-zero constant $a(E,h)\neq 0$ such that*

$$u_0(x) = a(E,h)\,\tilde u_0(x).$$

The connection coefficients $c_0^\pm(E,h)$ in (28) are analytic in E near $E_0>0$ and behave as

$$c_0^+(E,h) = a(E,h)(1+o(1)), \quad c_0^-(E,h) = -i\,a(E,h)(1+o(1))$$

uniformly for E near $E_0>0$ as $h\to 0$.

Proof: The first part is a direct consequence of (34) and the construction of $\tilde u_0(x)$.

The second part is an Airy type connection formula at least at the level of the principal term. Let us review briefly how to derive this by the Wronskian formulas (21) and (22). The coefficients c_0^+, c_0^- are given by

$$c_0^+ = a\frac{\mathcal{W}(\tilde u_0,u_0^-)}{\mathcal{W}(u_0^+,u_0^-)}, \quad c_0^- = -a\frac{\mathcal{W}(\tilde u_0,u_0^+)}{\mathcal{W}(u_0^+,u_0^-)}.$$

Since all the solutions involved just differ in the choice of the amplitude base point, we can apply formulas (21) and (22) to get

$$\mathcal{W}(u_0^+,u_0^-) = 2i\,w_+^{\mathrm{even}}(x_l^-;r_0,x_l^+), \qquad \mathcal{W}(\tilde u_0,u_0^-) = 2i\,w_+^{\mathrm{even}}(x_l^-;r_0,0).$$

The Wronskian $\mathcal{W}(\tilde u_0,u_0^+)$ is more delicate, since there is a branch cut between the origin and x_l^+, see Figure 2. Hence, u_0^+ should be considered on the other Riemann surface. Let us denote by $\hat x$ the point on the Riemann surface continued from x to the same point turning around r_0 by the angle -2π. Since $g_+(x)^{1/2} = -g_+(\hat x)^{1/2}$ and $g_+(x)^{1/4} = ig_+(\hat x)^{1/4}$, we have

$$z(x,r_0) = -z(\hat x,r_0), \quad H(x) = -iH(\hat x), \quad T_+(x) = i\,T_-(\hat x),$$

and for the series summands we get $w_{n,\pm}(\hat{x}) = w_{n,\mp}(x)$. Consequently, we have

$$u_0^+(x_+^l, r_0, 0) = iu_-(\hat{x}_+^l; r_0, 0),$$

which yields

$$\mathcal{W}(\tilde{u}_0, u_0^+) = -2\, w_+^{\text{even}}(\hat{x}_l^+; r_0, 0).$$

On the other hand, we know that

$$w_+^{\text{even}}(x_l^-; r_0; x_l^+) = 1 + O(h),$$

because we can take a path from x_l^+ to x_l^-, along which $x \mapsto \operatorname{Re} z(x)$ increases and which passes far away to the right from the turning point r_0. Hence

$$c_0^+ = a\, w_+^{\text{even}}(x_l^-; r_0; 0)\,(1 + O(h))\,, \quad c_0^- = -ia\, w_+^{\text{even}}(\hat{x}_l^+; r_0; 0)\,(1 + O(h))\,,$$

and it is enough for the proof to show

$$\lim_{h\to 0} w_+^{\text{even}}(x_l^-; r_0; 0) = \lim_{h\to 0} w_+^{\text{even}}(\hat{x}_l^+; r_0; 0) = 1. \tag{42}$$

Let $E > 0$. Recall that $w_{+,0}^{\text{even}} = \sum_{n=0}^{\infty}(I_0 \circ I_1)^n(1)$. Hence if we write $x_l^+ = iR$, then

$$\begin{aligned} w_+^{\text{even}}(\hat{x}_l^+; r_0; 0) &= w_{+,0}^{\text{even}}(iR) = 1 + (I_0 \circ I_1)(w_{+,0}^{\text{even}})|_{x=iR} \\ &= 1 + \frac{1}{2iE}\int_0^\infty \chi_{(0,ER/\nu)}(r)\frac{E + 3\nu^2 r^2/E^2}{1 + r^2(1 + \nu^2 r^2/E^3)^2} I_1(w_{+,0}^{\text{even}})|_{x=i\nu r/E}\, dr, \end{aligned}$$

where $\chi_{(0,ER/\nu)}$ is the characteristic function of the interval $(0, ER/\nu)$. The integrand function is dominated by an integrable function: indeed, Lemma 6.2 gives for $r \in (0, ER/\nu)$

$$\begin{aligned} |w_{+,0}^{\text{even}}(i\nu r/E)| &\le \sum_{n\ge 0} |w_{+,0}^{2n}(i\nu E/r)| \le \sum_{n\ge 0} \frac{K(r)^{2n}}{(2n)!}\left(\frac{r}{1+r}\right)^{2n} \le \cosh \tilde{K}, \\ |I_1(w_{+,0}^{\text{even}})(i\nu r/E)| &\le \tilde{K}\cosh \tilde{K}, \end{aligned}$$

with $\tilde{K} = 1 + 3R^2/E$, and

$$0 \le \chi_{(0,ER/\nu)}(r)\frac{E + 3\nu^2 r^2/E^2}{1 + r^2(1 + \nu^2 r^2/E^3)^2} \le \frac{E\tilde{K}}{1 + r^2}.$$

On the other hand, the integrand function tends to 0 as h tends to 0 (i.e. $\nu \to 0$) for any fixed $r > 0$, since $I_1(w_{+,0}^{\text{even}})|_{x=0} = 0$. By Lebesgue's dominated convergence theorem, we obtain the second identity of (42) for $E > 0$ and by analyticity for E near $E_0 > 0$. The first identity of (42) is obtained just in the same way. □

7. Connection formula near the critical point

In this section, we compute the matrix T_2, which connects $u_l^{\pm}(x) = u^{\pm}(x; r_1, x_l^{\pm})$ with $u_r^{\pm}(x) = u^{\pm}(x; r_2, x_r^{\pm})$,

$$(u_l^+(x), u_l^-(x)) = (u_r^+(x), u_r^-(x))T_2(E, h),$$

see (30). We will show

Proposition 7.1 *For all $E \in \mathbb{C}$ near $E_0 > 0$, the transfer matrix T_2 is of the form*

$$T_2(E, h) = \begin{pmatrix} t(E,h) & s(E,h) \\ -\overline{s(E,h)} & -\overline{t(E,h)} \end{pmatrix}, \tag{43}$$

and the asymptotics of $t(E,h)$ and $s(E,h)$ for $h \to 0$ are given by

$$t(E,h) = -\sqrt{\frac{\pi h}{2}}\,\tilde{\nu}\, E^{-3/4}\, e^{-i\pi/4} + O(h|\ln h|), \qquad s(E,h) = i + O(h).$$

The proof of Proposition 7.1 is given in Section 7.5.

Remark 7.2 From the symmetry properties

$$u_r^+ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{u_r^-}, \qquad u_l^- = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{u_l^+}, \tag{44}$$

we know *a priori* that T_2 is indeed of the claimed form (43). Hence, it remains to prove the asymptotic behaviour of $t(E,h)$ and $s(E,h)$ as $h \to 0$. ◇

The exact WKB method is not enough to compute the h-asymptotics of $T_2(E,h)$, because the two turning points r_1 and r_2 tend to the same point $x = \sqrt{E}$ as $h \to 0$. In other words: The Wronskian of two exact WKB solutions, say u_l^+ and u_r^-, is given in terms of w_+^{even} computed along a path from x_l^+ to x_r^- passing between r_1 and r_2, but the h-asymptotic formula for w_+^{even} (Proposition 3.3) fails to hold because of the singularity of the function H at r_1 and r_2.

Hence, one resorts to a microlocal study of the equation $P_\nu u = Eu$ near the point $(x,\xi) = (\sqrt{E},\xi)$ for $E > 0$, where ξ is the dual variable of x. The equation $P_\nu u = Eu$ is reduced to a simple microlocal normal form $Qw = 0$ (see Section 7.1), whose solutions are well studied. From these solutions we obtain two basis sets of microlocal solutions $(\tilde{f}^+, \tilde{f}^-)$, $(\tilde{g}^+, \tilde{g}^-)$ of $P_\nu u = Eu$, which are related via a constant matrix R:

$$(\tilde{f}^+, \tilde{f}^-)R = (\tilde{g}^+, \tilde{g}^-)$$

with

$$R = \begin{pmatrix} p & q \\ -q & -p \end{pmatrix},$$

see Section 7.2. The exact WKB solutions $u_l^\pm$, $u_r^\pm$ are expressed in terms of these basis sets by

$$(u_l^+, u_l^-) = (\tilde{f}^+, \tilde{f}^-)A_l = (\tilde{g}^+, \tilde{g}^-)B_l,$$

$$(u_r^+, u_r^-) = (\tilde{f}^+, \tilde{f}^-)A_r = (\tilde{g}^+, \tilde{g}^-)B_r,$$

where the constant matrices $A_{l,r}$ and $B_{l,r}$ satisfy

$$A_l = RB_l, \quad A_r = RB_r. \tag{45}$$

Then the matrix T_2 is given by

$$T_2 = A_r^{-1}A_l = B_r^{-1}B_l. \tag{46}$$

Hence, the h-asymptotics of $t(E,h)$ and $s(E,h)$ can be obtained from the study of the connection matrices $A_{l,r}$ and $B_{l,r}$ (see Sections 7.3 and 7.4).

7.1. **Normal form.** We now transform the equation $(P_\nu - E)u = 0$ near $(x,\xi) = (\sqrt{E},\xi)$, $E > 0$, to a simple microlocal normal form $Qw = 0$.

Theorem 7.3 *Let $E > 0$ and $u(x,h)$ be a solution of $(P_\nu - E)u = 0$. Let V be the metaplectic operator associated with the $\frac{\pi}{4}$-rotation in phase space*

$$\kappa_{\frac{\pi}{4}} : T^*\mathbb{R} \to T^*\mathbb{R}, \quad (x,\xi) \mapsto \tfrac{1}{\sqrt{2}}(x-\xi, x+\xi).$$

There exists a locally diffeomorphic change of coordinates $x \mapsto \phi(x) = y$, $x > 0$, with $\phi(\sqrt{E}) = 0$ and a matrix-valued $\mathcal{C}^\infty$-symbol $M(y,h) = \mathrm{Id} + O(h)$ such that for any cut-off function $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$ identically equal to 1 in an interval around $y = 0$

$$w(y,h) = V\big(\chi(y) M(y,h) u(\phi^{-1}(y),h)\big)$$

satisfies $Qw = r$, where

$$Q = \begin{pmatrix} y & \frac{\gamma}{\sqrt{2}} \\ -\frac{\overline{\gamma}}{\sqrt{2}} & -hD_y \end{pmatrix},$$

$\gamma = \gamma(E,h)$ is a constant with $\gamma(E,h) = \frac{\tilde\nu}{\sqrt{2}} E^{-3/4} h + O(h^2)$, and $r(y,h) = O(h^\infty)$ uniformly in an interval around $y = 0$ together with all its derivatives.

Remark 7.4 The terminology *metaplectic operator* will be recalled in Appendix D. $\diamond$

Proof of Theorem 7.3: We proceed in three steps to reduce the equation $P_\nu u = Eu$, that is

$$hD_x u(x) = A(x)u(x), \qquad A(x) = \begin{pmatrix} x^2 - E & \nu/x \\ -\nu/x & E - x^2 \end{pmatrix}. \tag{47}$$

First step. One turns the quadratic diagonal entries of $A(x)$ into linear ones. Let $y = \phi(x)$ with

$$\phi(x) = (x - \sqrt{E})\left(\tfrac{2}{3}(x-\sqrt{E}) + 2\sqrt{E}\right)^{1/2}.$$

In the complex right half-plane, the function $\phi(x)$ is a biholomorphic map with $\phi(\sqrt{E}) = 0$ and $\phi(x)\phi'(x) = x^2 - E$. The function

$$\psi(y) = \psi(\phi(x)) = \frac{(\frac{2}{3}(x-\sqrt{E}) + 2\sqrt{E})^{1/2}}{x(x+\sqrt{E})} \tag{48}$$

is analytic in a neighborhood of $y = 0$, satisfying $\psi(0) = \frac{E^{-3/4}}{\sqrt{2}}$ and $\psi(\phi(x))\phi'(x) = 1/x$. Moreover, if $u(x)$ satisfies (47), then $v(y) = v(\phi(x)) = u(x)$ satisfies

$$hD_y v(y) = \begin{pmatrix} y & \nu\psi(y) \\ -\nu\psi(y) & -y \end{pmatrix} v(y).$$

Second step. Recall that $\nu = \tilde\nu h$ with $\tilde\nu \in \mathbb{N} - \frac{1}{2}$. The second step makes the off-diagonal entries constant modulo $O(h^\infty)$ by a change of the unknown function

$$\widetilde{w}(y,h) = M(y,h)v(y,h).$$

Lemma D.1 constructs a matrix-valued $\mathcal{C}^\infty$-symbol $M(y,h) = \mathrm{Id} + O(h)$ such that $\widetilde{w}(y,h)$ satisfies

$$\begin{pmatrix} hD_y - y & -\gamma \\ \overline{\gamma} & hD_y + y \end{pmatrix} \widetilde{w}(y,h) = r(y,h)\widetilde{w}(y,h) \tag{49}$$

where $\gamma = \frac{\tilde{\nu}}{\sqrt{2}} E^{-3/4} h + O(h^2)$ and $r(y,h) = O(h^\infty)$ uniformly in an interval around $y = 0$ together with all its derivatives.

Third step. Multiplying a cut off function χ and then operating the metaplectic operator V from the left to equation (49), we obtain by Lemma D.2

$$Qw(y,h) = -\frac{1}{\sqrt{2}} V \left(\chi(y) r(y,h) \widetilde{w}(y,h) - ih\chi'(y)\widetilde{w}(y,h) \right).$$

The right hand side is of $O(h^\infty)$ uniformly in interval around $y = 0$ together with its all derivatives. □

Remark 7.5 The proof of Theorem 7.3 shows, that γ, ϕ, and M depend analytically on E for $E \in \mathbb{C}$ near some $E_0 > 0$. ◇

Remark 7.6 In Appendix E we prove, that the $M_n(y)$ with $M(y,h) \sim \sum_{n=0}^{\infty} M_n(y) h^n$ have an analytic resummation, which is a Gevrey symbol of index 2. Hence, the second scale $\sqrt{h}$, on which the two-scale Wigner measures in [11] are based, reappears naturally in our normal form transformation. ◇

7.2. **Solutions of the normal form.** Here we compute the solutions of the normal form. The equation

$$Qw = 0, \quad w = {}^t(w_1, w_2)$$

is equivalent to

$$w_1 = -\frac{\gamma}{\sqrt{2}y} w_2\,, \quad \frac{h}{i} y w_2' = \frac{|\gamma|^2}{2} w_2\,. \tag{50}$$

This is a well-studied saddle point problem (see for example [19], [7], or section 5 in [29]). The system (50) has two maximal solutions

$$f^\pm(y) = {}^t(f_1^\pm(y), f_2^\pm(y)),$$

with

$$f_1^\pm(y) = -\frac{\gamma}{\sqrt{2}y} \chi_{(0,\infty)}(\pm y)\, |y|^{\frac{i}{2h}|\gamma|^2}\,,$$

$$f_2^\pm(y) = \chi_{(0,\infty)}(\pm y)\, |y|^{\frac{i}{2h}|\gamma|^2}\,,$$

where $\chi_{(0,\infty)}$ is the characteristic function of the interval $(0, \infty)$. Moreover, we have two additional solutions

$$g^\pm(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h f^\pm(y), \tag{51}$$

where $\mathcal{F}_h u(\eta) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-iy\eta/h} u(y) dy$ is the h-Fourier transform and $\mathcal{C}u = \bar{u}$ is the complex conjugate. Indeed, we have the following identity:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h Q = -Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h.$$

Since $\mathcal{C}\mathcal{F}_h = \mathcal{F}_h^{-1}\mathcal{C}$, we also have

$$f^\pm(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h g^\pm(y). \tag{52}$$

These four solutions are linearly dependent.

Proposition 7.7 *The solutions $f^{\pm}$ and $g^{\pm}$ of $Qw = 0$ are connected by*

$$(g^+, g^-) = (f^+, f^-)R, \qquad R = \begin{pmatrix} p & q \\ -q & -p \end{pmatrix}, \tag{53}$$

where

$$\begin{aligned} p &= \frac{h^{\frac{1}{2}-\frac{i}{2h}|\gamma|^2}}{i\gamma\sqrt{\pi}}\,\Gamma(1-\tfrac{i}{2h}|\gamma|^2)\exp(\frac{\pi}{4h}|\gamma|^2), \\ q &= \frac{h^{\frac{1}{2}-\frac{i}{2h}|\gamma|^2}}{i\gamma\sqrt{\pi}}\,\Gamma(1-\tfrac{i}{2h}|\gamma|^2)\exp(-\frac{\pi}{4h}|\gamma|^2). \end{aligned}$$

Proof: The proof is just the one of Proposition 5.5 in [29]. We check $g_1^+ = pf_1^+ - qf_1^-$. One writes

$$g_1^+(y) = \mathcal{C}\mathcal{F}_h f_2^+(y) = \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{2\pi h}} \int_0^\infty e^{i(y+i\epsilon)\eta/h}\,\eta^{-\frac{i}{2h}|\gamma|^2} d\eta$$

and substitutes $i(y+i\epsilon)\eta/h = -t$ to obtain

$$g_1^+(y) = \frac{h^{1/2-\frac{i}{2h}|\gamma|^2}}{\sqrt{2\pi}} \lim_{\epsilon\to 0^+} (\epsilon - iy)^{\frac{i}{2h}|\gamma|^2-1} \int_{\alpha(\epsilon)} e^{-t}\,t^{-\frac{i}{2h}|\gamma|^2} dt,$$

where $\alpha(\epsilon)$ is the image of the interval $(0,\infty)$ under the map $\eta \mapsto (\epsilon - iy)\eta/h$. Using Euler's integral form of the gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, $\mathrm{Re}(z) > 0$, and Cauchy's integral formula, one gets

$$g_1^+(y) = \frac{h^{1/2-\frac{i}{2h}|\gamma|^2}}{\sqrt{2\pi}} (-iy)^{\frac{i}{2h}|\gamma|^2-1}\,\Gamma(1-\tfrac{i}{2h}|\gamma|^2) = pf_1^+(y) - qf_1^-(y).$$

□

Remark 7.8 In view of the relations (51) and (52), the matrix R satisfies $R\bar{R} = \mathrm{Id}$, that is $|p|^2 - |q|^2 = 1$ and $p\bar{q} = \bar{p}q$. ◇

7.3. **Frequency sets of the microlocal and WKB solutions.** Let us study the frequency set of the microlocal solutions $f^{\pm}$, $g^{\pm}$ and the exact WKB solutions $u_{\pm}^l$, $u_{\pm}^r$.

First, the frequency set of the microlocal solutions are subsets of $\mathbb{R}_y \times \mathbb{R}_\eta$ as follows:

$$\begin{aligned} \mathrm{FS}(f_1^{\pm}) &\subset \{y=0\} \cup \{\eta = 0, \pm y > 0\}, \\ \mathrm{FS}(g_1^{\pm}) &\subset \{\eta=0\} \cup \{y = 0, \pm \eta > 0\}. \end{aligned}$$

Second, let $\sigma_{r,l}^{\pm} \subset \mathbb{R}_x \times \mathbb{R}_\xi$ be the Lagrangian manifolds defined by

$$\sigma_l^{\pm} = \{r_0 < x < r_1,\ \xi = \pm(E - x^2)\}, \qquad \sigma_r^{\pm} = \{r_2 < x,\ \xi = \pm(x^2 - E)\}. \tag{54}$$

Since $z'(x) = \sqrt{\nu^2 - x^2(E-x^2)^2}/x$, one has by Lemma C.4

$$\mathrm{FS}(u_l^{\pm}) \cap \{r_0 < x < r_1\} \subset \sigma_l^{\pm}, \qquad \mathrm{FS}(u_r^{\pm}) \cap \{x > r_2\} \subset \sigma_r^{\pm}.$$

Now we transform to the normal form of Theorem 7.3, that is operate N to the exact WKB solutions $u_{l,r}^{\pm}$,

$$Nu(y,h) = V\big(\chi(y)M(y,h)u(\phi^{-1}(y),h)\big).$$

Since $u_{l,r}^{\pm}$ are solutions of the equation $(P_\nu - E)u = 0$, we see that for any $k \geq 0$

$$D_y^k Q(Nu_{l,r}^{\pm}) = O(h^\infty).$$

Hence, $Nu_{l,r}^{\pm}$ are microlocal solutions of $Qw = 0$ near $(0,0)$. Since the vector space of such solutions is two-dimensional, see Proposition 17 of [7], there exist matrix-valued $\mathcal{C}^\infty$-symbols

$$A_l = (\alpha_{jk}^l), \quad A_r = (\alpha_{jk}^r), \quad B_l = (\beta_{jk}^l), \quad B_r = (\beta_{jk}^r), \tag{55}$$

such that microlocally near $(0,0)$

$$\begin{aligned} (Nu_l^+, Nu_l^-) &= (f^+, f^-)A_l = (g^+, g^-)B_l, \\ (Nu_r^+, Nu_r^-) &= (f^+, f^-)A_r = (g^+, g^-)B_r. \end{aligned}$$

Returning back to the (x,ξ) variables, i.e. operating N^{-1} from the left, we have microlocally near $(\sqrt{E}, 0)$,

$$\begin{aligned} (u_l^+, u_l^-) &= (\tilde{f}^+, \tilde{f}^-)A_l = (\tilde{g}^+, \tilde{g}^-)B_l, \\ (u_r^+, u_r^-) &= (\tilde{f}^+, \tilde{f}^-)A_r = (\tilde{g}^+, \tilde{g}^-)B_r, \end{aligned}$$

where

$$\tilde{f}^{\pm} = N^{-1}(\tilde{\chi} f^{\pm}), \quad \tilde{g}^{\pm} = N^{-1}(\tilde{\chi} g^{\pm}) \tag{56}$$

and $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R})$ a cut-off function, which is identically equal to 1 near $y = 0$ and satisfies $\mathrm{supp}(\tilde{\chi}) \subset \mathrm{supp}(\chi)$.

By Lemma D.2, we have

$$\mathrm{FS}(\tilde{f}^{\pm}) = \kappa_\phi^{-1}\big(\kappa_{\frac{\pi}{4}} \mathrm{FS}(\tilde{\chi} f^{\pm})\big), \quad \mathrm{FS}(\tilde{g}^{\pm}) = \kappa_\phi^{-1}\big(\kappa_{\frac{\pi}{4}} \mathrm{FS}(\tilde{\chi} g^{\pm})\big),$$

where

$$\kappa_\phi^{-1} : T^*\mathbb{R} \to T^*\mathbb{R}, \quad (x,\xi) \mapsto \big(\phi^{-1}(x), \xi\, \phi'(\phi^{-1}(x))\big)$$

is the inverse of the canonical transformation $\kappa_\phi(x,\xi) = (\phi(x), \xi/\phi'(x))$ associated with ϕ. Clearly,

$$\begin{aligned} \kappa_{\frac{\pi}{4}}\, \mathrm{FS}(f_1^{\pm}) &\subset \{\eta = -y\} \cup \{\eta = y,\ \pm y > 0\}, \\ \kappa_{\frac{\pi}{4}}\, \mathrm{FS}(g_1^{\pm}) &\subset \{\eta = y\} \cup \{\eta = -y,\ \mp y > 0\}. \end{aligned}$$

Since $\phi(x)\phi'(x) = x^2 - E$, this yields

$$\begin{aligned} &\mathrm{FS}(\tilde{f}^+) \cap U \subset \sigma_l^- \cup \sigma_r^+ \cup \sigma_r^-, \qquad && \mathrm{FS}(\tilde{f}^-) \cap U \subset \sigma_l^+ \cup \sigma_l^- \cup \sigma_r^+, \\ &\mathrm{FS}(\tilde{g}^+) \cap U \subset \sigma_l^+ \cup \sigma_r^+ \cup \sigma_r^-, && \mathrm{FS}(\tilde{g}^-) \cap U \subset \sigma_l^+ \cup \sigma_l^- \cup \sigma_r^- \end{aligned}$$

with $U = \{r_0 < x < r_1 \text{ or } x > r_2\}$, see Figure 3. σ_l^+ is neither contained in $\mathrm{FS}(u_l^-)$ nor in $\mathrm{FS}(\tilde{f}^+)$, while it is contained in $\mathrm{FS}(\tilde{f}^-)$. Hence, $\alpha_{22}^l = 0$. Analogously one obtains

$$\alpha_{11}^r = \alpha_{22}^l = \beta_{12}^r = \beta_{21}^l = 0. \tag{57}$$

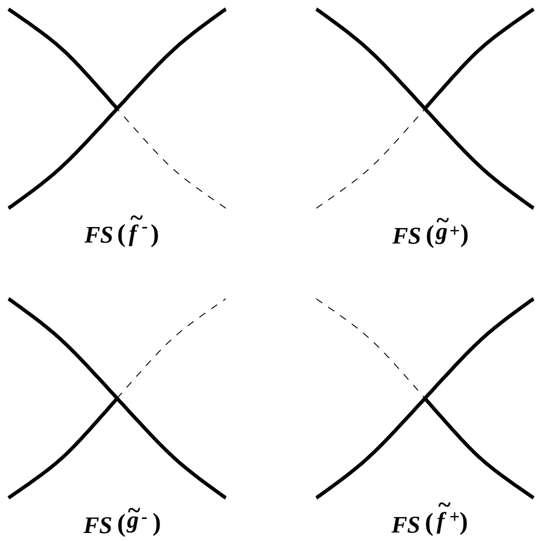

FIGURE 3. Frequency sets of the microlocal solutions $\tilde{f}^{\pm}$ and $\tilde{g}^{\pm}$ defined in (56).

7.4. **Connection between microlocal and exact WKB solutions.** We now compute the remaining coefficients of the matrices $A_{l,r}$ and $B_{l,r}$ connecting the microlocal solutions $\tilde{f}^{\pm}$, $\tilde{g}^{\pm}$ with the WKB solutions $u_l^{\pm}$, $u_r^{\pm}$ of $P_\nu u = Eu$, $E > 0$.

As a first step, the stationary phase method gives the following formulae for the microlocal solutions, whose proof is to be found in Appendix F.

Lemma 7.9 *For the microlocal solutions $\tilde{f}^{\pm}$ of $P_\nu u = Eu$, $E > 0$, defined in (56), one has microlocally near the Lagrangian manifolds σ_l^+ and σ_r^- defined in (54)*

$$\tilde{f}^-(x) = e^{i\pi/8}\, 2^{1/4} e^{+z(x;r_1)/h} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + O(h)) \quad \text{near } \sigma_l^+, \tag{58}$$

$$\tilde{f}^+(x) = e^{i\pi/8}\, 2^{1/4} e^{-z(x;r_2)/h} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + O(h)) \quad \text{near } \sigma_r^-. \tag{59}$$

It remains to connect $e^{\pm z(x;r_1)/h}$ and $e^{\pm z(x;r_2)/h}$ to the exact WKB solutions $u_l^{\pm}$ and $u_r^{\pm}$.

Proposition 7.10 *With the notation of Lemma 7.9, we have microlocally*

$$u_l^+ = k_l^+ \tilde{f}^- \quad \text{near } \sigma_l^+, \quad u_r^+ = k_r^+ \tilde{g}^+ \quad \text{near } \sigma_r^+,$$

$$u_l^- = k_l^- \tilde{g}^- \quad \text{near } \sigma_l^-, \quad u_r^- = k_r^- \tilde{f}^+ \quad \text{near } \sigma_r^-,$$

where

$$k_l^+ = -2^{3/4} e^{i\pi/8} (1 + O(h)), \qquad k_r^- = -2^{3/4} e^{-i3\pi/8} (1 + O(h)),$$

and

$$k_l^- = -i\, \overline{k_l^+}, \qquad k_r^+ = i\, \overline{k_r^-}.$$

Proof: To prove these relations, say the first one, it is enough to calculate the asymptotic behavior of $\tilde{f}^-$ and u_l^+ near σ_l^+. First, recall the exact WKB formula

$$u_l^\pm(x) = e^{\pm z(x;r_1)/h}\, T_\pm(z(x;r_1)) \begin{pmatrix} w_\pm^{\mathrm{even}}(x) \\ w_\pm^{\mathrm{odd}}(x) \end{pmatrix} = e^{\pm z(x;r_1)/h}\, T_\pm(z(x;r_1)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+O(h))$$

with

$$T_\pm(z) = \begin{pmatrix} H^{-1}(z) \mp iH(z) & H^{-1}(z) \pm iH(z) \\ -H^{-1}(z) \mp iH(z) & -H^{-1}(z) \pm iH(z) \end{pmatrix}$$

and

$$H(z(x)) = \left(\frac{\nu + Ex - x^3}{\nu - Ex + x^3} \right)^{1/4}.$$

Let $x \in]r_0, r_1[$ be fixed. With the branches of the fourth root chosen in Section 4, one has $H(z(x)) = e^{-i\pi/4} + O(h)$ as $h \to 0$. Hence,

$$\begin{aligned} T_+(z(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} H^{-1}(z(x)) - iH(z(x)) \\ -H^{-1}(z(x)) - iH(z(x)) \end{pmatrix} = -2\, e^{i\pi/4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+O(h)), \\ T_-(z(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} H^{-1}(z(x)) + iH(z(x)) \\ -H^{-1}(z(x)) + iH(z(x)) \end{pmatrix} = +2\, e^{i\pi/4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+O(h)), \end{aligned}$$

and therefore

$$\begin{aligned} u_l^+(x) &= -2\, e^{i\pi/4} e^{+z(x;r_1)/h} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+O(h)), \\ u_l^-(x) &= +2 e^{i\pi/4} e^{-z(x;r_1)/h} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+O(h)). \end{aligned} \tag{60}$$

Comparing (60) and (58), we immediately obtain

$$k_l^+ = -2^{3/4} e^{i\pi/8} (1+O(h)).$$

Similarly, for fixed x in the interval $]r_2, \infty[$, $H(z(x)) = e^{\pi i/4} + O(h)$ as $h \to 0$, and

$$\begin{aligned} u_r^+(x) &= +2 e^{-i\pi/4} e^{+z(x;r_2)/h} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+O(h)), \\ u_r^-(x) &= -2 e^{-i\pi/4} e^{-z(x;r_2)/h} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+O(h)). \end{aligned} \tag{61}$$

Comparing (61) and (59), we immediately get

$$k_r^- = -2^{3/4} e^{-i3\pi/8} (1+O(h)).$$

For the computation of k_r^+ and k_l^- we use some symmetry properties. Recall that

$$g^\pm(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h f^\pm(y),$$

and

$$\mathcal{C}\mathcal{F}_h = \mathcal{F}_h^{-1}\mathcal{C}, \quad \mathcal{C}V^{-1} = V\mathcal{C}, \quad V^{-1}\mathcal{F}_h^{-1} = V,$$

where the last identity comes from $\mathcal{F}_h^{-1}$ being the metaplectic operator of the transformation $(x,\xi) \mapsto (\xi, -x)$, up to a normalizing constant. Hence,

$$V^{-1}g^\pm = V^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}\mathcal{F}_h f^\pm = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V^{-1}\mathcal{F}_h^{-1}\mathcal{C} f^\pm = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}V^{-1} f^\pm$$

and

$$\tilde{g}^{\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\tilde{f}^{\pm}}. \tag{62}$$

On the other hand,

$$u_r^+ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{u_r^-}, \qquad u_l^- = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{u_l^+},$$

which together with (62) yields

$$k_l^- = -i\, \overline{k_l^+}, \qquad k_r^+ = i\, \overline{k_r^-}.$$

□

Proposition 7.10 yields for the connection matrices $A_{l,r}$ and $B_{l,r}$ defined in (55)

$$\alpha_{21}^l = k_l^+, \quad \alpha_{12}^r = k_r^-, \quad \beta_{22}^l = k_l^-, \quad \beta_{11}^r = k_r^+.$$

Combining these with the knowledge on the vanishing matrix elements (57) and the relations $A_l = RB_l$, $A_r = RB_r$, one obtains

$$B_l = \begin{pmatrix} -\frac{1}{q} k_l^+ & -\frac{p}{q} k_l^- \\ 0 & k_l^- \end{pmatrix}, \qquad B_r = \begin{pmatrix} k_r^+ & 0 \\ -\frac{p}{q} k_r^+ & \frac{1}{q} k_r^- \end{pmatrix}. \tag{63}$$

7.5. **Computation of $T_2(E,h)$.** Finally, we prove Proposition 7.1 calculating the asymptotic behavior of $t(E,h)$ and $s(E,h)$ as $h \to 0$.

Let $E > 0$. By (46) and (63),

$$T_2 = B_r^{-1} B_l = \begin{pmatrix} -\dfrac{1}{q} \dfrac{k_l^+}{k_r^+} & -\dfrac{p}{q} \dfrac{k_l^-}{k_r^+} \\ -\dfrac{p}{q} \dfrac{k_l^+}{k_r^-} & \dfrac{q^2 - p^2}{q} \dfrac{k_l^-}{k_r^-} \end{pmatrix}. \tag{64}$$

Remark 7.11 The identity (64) is consistent with $T_2 = \begin{pmatrix} t & s \\ -\overline{s} & -\overline{t} \end{pmatrix}$, since

$$\overline{\left(\frac{k_l^+}{k_r^+} \right)} = -\frac{k_l^-}{k_r^-}, \qquad \overline{\left(\frac{k_l^+}{k_r^-} \right)} = -\frac{k_l^-}{k_r^+}, \qquad \overline{\left(\frac{p}{q} \right)} = \frac{p}{q}, \qquad \frac{1}{\overline{q}} = \frac{p^2 - q^2}{q},$$

which is checked by direct calculation. ◇

By Proposition 7.7 and Proposition 7.10, we then get

$$t = -\frac{1}{q} \frac{k_l^+}{k_r^+} = -\frac{\sqrt{\pi}\, \gamma\, \exp\left(\frac{\pi}{4h} |\gamma|^2 + \frac{i}{2h} |\gamma|^2 \ln h \right)}{\sqrt{h}\, \Gamma(1 - \frac{i}{2h} |\gamma|^2)\, e^{i\pi/4}} \left(1 + O(h)\right).$$

Since $\gamma = \frac{\tilde{\nu}}{\sqrt{2}} E^{-3/4} h + O(h^2)$ and

$$\Gamma(1 - \tfrac{i}{2h} |\gamma|^2) = \sqrt{\frac{\frac{\pi}{2h} |\gamma|^2}{\sin(\frac{\pi}{2h} |\gamma|^2)} \left(1 + \frac{|\gamma|^2}{4h^2} \right)} = 1 + O(h),$$

we have

$$t = -\sqrt{\frac{\pi h}{2}}\, \tilde{\nu}\, E^{-3/4}\, e^{-i\pi/4} + O(h |\ln h|)$$

and

$$s = -\frac{p}{q}\frac{k_l^-}{k_r^+} = -i\,\exp\left(\tfrac{\pi}{2h}|\gamma|^2\right)(1+O(h)) = -i + O(h).$$

Since all the terms involved depend analytically on E for $E \in \mathbb{C}$ near $E_0 > 0$, see Remark 7.5, we have proven Proposition 7.1.

8. Proof of the main results

In this section, we compute the Bohr-Sommerfeld type quantization condition of Theorem 1.2 and derive the semiclassical distribution of resonances given in Theorem 1.3.

8.1. **Quantization condition.** Recall that Proposition 4.1 gives the quantization condition of resonances as $c^+(E,h) = 0$, where $c^{\pm}(E,h)$ is the product of three transfer matrices T_1, T_2, T_3 and the connection coefficients $c_0^{\pm}$,

$$\begin{pmatrix} c^+ \\ c^- \end{pmatrix} = T_3 T_2 T_1 \begin{pmatrix} c_0^+ \\ c_0^- \end{pmatrix},$$

see identity (32). On the other hand, we have calculated the following asymptotics:

$$\begin{aligned}
T_1 &= \begin{pmatrix} e^{S_{01}/h} & 0 \\ 0 & e^{-S_{01}/h} \end{pmatrix}, \\
T_2 &= \begin{pmatrix} t & -i+O(h) \\ -i+O(h) & -\bar{t} \end{pmatrix}, \quad t = -\sqrt{\frac{\pi h}{2}}e^{-i\pi/4}\tilde{\nu}E^{-3/4} + O(h|\ln h|), \\
T_3 &= 2e^{-i\pi/4}\begin{pmatrix} e^{S_{2\infty}/h}(1+O(h)) & O(e^{-\delta/h}) \\ O(e^{-\delta/h}) & e^{-S_{2\infty}/h}(1+O(h)) \end{pmatrix}, \\
\begin{pmatrix} c_0^+ \\ c_0^- \end{pmatrix} &= a\begin{pmatrix} 1+o(1) \\ -i+o(1) \end{pmatrix},
\end{aligned}$$

see (33), Proposition 7.1, Proposition 5.3, and Proposition 6.3, respectively. Then,

$$c^+ = 2a\,e^{-\pi i/4}\,e^{S_{2\infty}/h}(1+O(h))\left(t\,e^{S_{01}/h}(1+o(1)) - e^{-S_{01}/h}(1+o(1))\right) + O(e^{-\delta/h}).$$

Hence, $c^+(E,h) = 0$ if and only if

$$\sqrt{\frac{\pi h}{2}}\,\tilde{\nu}\,e^{-i\pi/4}\,E^{-3/4}\,e^{2S_{01}(E,h)/h} + 1 = o(1) \tag{65}$$

as $h \to 0$, which proves Theorem 1.2.

8.2. **Distribution of resonances.** We now study the asymptotic behavior of the function $S_{01}(E,h)$ as $h \to 0$. Recall that

$$S_{01}(E,h) = \int_{r_0}^{r_1} \frac{\sqrt{\nu^2 - r^2(E-r^2)^2}}{r}\,dr.$$

If $\nu = \tilde{\nu}h > 0$ is sufficiently small and $E > 0$, then $\nu^2 - r^2(E-r^2)^2 > 0$ for $r \in (r_0, r_1)$, and the square root in the formula for $S_{01}(E,h)$ is taken to be positive. Substituting $y = r^2/E$, one gets

$$S_{01}(E,h) = iE^{3/2}\int_{y_0}^{y_1} \frac{\sqrt{y(1-y)^2 - \nu^2/E^3}}{2y}\,dy$$

with $y_0 = r_0^2/E$ and $y_1 = r_1^2/E$.

y_0 and y_1 are zeros of the cubic polynomial $y(1-y)^2 - \mu^2$ with $\mu = \nu E^{-3/2}$. If $\mu > 0$ is small and positive, then $y(1-y)^2 - \mu^2$ has three zeros $0 < y_0(\mu) < y_1(\mu) < 1 < y_2(\mu)$ with $y_0(\mu) \to 0$ and $y_{1,2}(\mu) \to 1$ as $\mu \to 0$. We define

$$I(\mu) = \int_{y_0(\mu)}^{y_1(\mu)} \frac{\sqrt{y(1-y)^2 - \mu^2}}{2y}\, dy,$$

where the square root is taken to be positive for $0 < \mu \ll 1$. Since

$$S_{01}(E,h) = iE^{3/2} I(\mu), \qquad \mu = \frac{\nu}{E^{3/2}} = \frac{\tilde{\nu}}{E^{3/2}} h\,,$$

we study the asymptotic behavior of the function $I(\mu)$ as $\mu \to 0$. For this, we have to understand the μ-dependance of $y_0(\mu)$ and $y_1(\mu)$. When μ^2 turns around 0 once in the positive sense (i.e. μ becomes $e^{\pi i}\mu$), then $y_0(\mu)$ turns around 0 in the positive sense and $y_1(\mu)$ and $y_2(\mu)$ exchange their position turning half around 1 in the positive sense. As a consequence, taking the branch into account,

$$I(e^{\pi i}\mu) = I(\mu) + R(\mu) + T(\mu) \tag{66}$$

with

$$R(\mu) = -i \int_{\Gamma_0} \frac{\sqrt{\mu^2 - y(1-y)^2}}{2y} dy, \quad T(\mu) = -i \int_{y_1(\mu)}^{y_2(\mu)} \frac{\sqrt{\mu^2 - y(1-y)^2}}{2y} dy,$$

where Γ_0 is a contour around 0. These functions have the following properties:

Lemma 8.1 *$T(\mu)$ is a holomorphic function of μ^2 at $\mu = 0$, and*

$$R(\mu) = -\pi\mu, \quad T(\mu) = \frac{\pi i \mu^2}{4}(1 + O(\mu^2)).$$

In particular, $R(e^{\pi i}\mu) = -R(\mu)$ and $T(e^{\pi i}\mu) = T(\mu)$.

Proof: By the residue theorem, $R(\mu) = \pi\sqrt{\mu^2} = -\pi\mu$. For the study of $T(\mu)$, we move by the locally biholomorphic change of variables $v = \sqrt{y}(y-1)$ from a neighborhood of $y = 1$ to a neighborhood of $v = 0$,

$$T(\mu) = -i \int_{-\mu}^{\mu} \sqrt{\mu^2 - v^2}\, f(v)\, dv$$

where $f(v) = (2y(v))^{-1} \frac{d}{dv} y(v)$ is holomorphic in a neighborhood of $v = 0$ and satisifies $f(v) = \frac{1}{2} + f'(0)v + O(v^2)$. Hence,

$$T(\mu) = i\mu^2 \int_{-1}^{1} \sqrt{1 - w^2}\, f(\mu w)\, dw = i\mu^2 \left(\tfrac{1}{2} \int_{-1}^{1} \sqrt{1 - w^2}\, dw + O(\mu^2) \right),$$

since $\int_{-1}^{1} \sqrt{1 - w^2}\, w\, dw = 0$. □

Proposition 8.2 *$I(\mu)$ is ramified at $\mu = 0$ and satisfies*

$$I(\mu) = \tfrac{2}{3} + \tfrac{\pi}{2}\mu + O(\mu^2 |\ln \mu|) \qquad (\mu \to 0).$$

Proof: By Lemma 8.1, we have from (66)

$$\begin{aligned} I(e^{2\pi i}\mu) &= I(e^{\pi i}\mu) + R(e^{\pi i}\mu) + T(e^{\pi i}\mu) = (I(\mu) + R(\mu) + T(\mu)) - R(\mu) + T(\mu) \\ &= I(\mu) + 2T(\mu). \end{aligned}$$

Since $\ln(e^{2\pi i}\mu) = \ln \mu + 2\pi i$, this means that the function

$$A(\mu) = I(\mu) - \tfrac{1}{\pi i}\, T(\mu) \ln \mu$$

is single-valued around $\mu = 0$. Moreover, since $T(\mu)$ behaves quadratically in μ near $\mu = 0$, $A(\mu)$ is holomorphic near $\mu = 0$ with

$$A(\mu) \overset{\mu \to 0}{\longrightarrow} A(0) = I(0) = \int_0^1 (1 - x^2) dx = \tfrac{2}{3}.$$

Differentiating equation (66), one gets $A'(0) = I'(0) = \frac{\pi}{2}$. Hence,

$$\begin{aligned} I(\mu) &= A(0) + A'(0)\mu + \tfrac{1}{\pi i}\, T(\mu) \ln \mu + O(\mu^2) = \tfrac{2}{3} + \tfrac{1}{\pi i}\, T(\mu) \ln \mu + \tfrac{\pi}{2}\mu + O(\mu^2) \\ &= \tfrac{2}{3} + \tfrac{\pi}{2}\mu + O(\mu^2 |\ln \mu|). \end{aligned}$$

□

Proof of Theorem 1.3: The quantization condition (65) is satisfied, if and only if there exists an integer $k \in \mathbb{Z}$ such that

$$2S_{01}(E,h) - \tfrac{1}{2}\, h \ln E^{3/2} + \tfrac{1}{2}\, h \ln\left(\tfrac{\pi}{2}\tilde{\nu}^2 h\right) = \left(2k + \tfrac{5}{4}\right) i\pi h + o(h). \tag{67}$$

Setting $\lambda = E^{3/2}$, Proposition 8.2 implies

$$S_{01}(E,h) = i\lambda\, I(\tilde{\nu}h/\lambda) = \tfrac{2}{3}\, i\lambda + \tfrac{\pi}{2}\, i\tilde{\nu}h + O(h^2 |\ln h|),$$

and the quantization condition (67) becomes

$$\tfrac{4}{3} i\lambda - \tfrac{1}{2}\, h \ln \lambda + \tfrac{1}{2}\, h \ln\left(\tfrac{\pi}{2}\tilde{\nu}^2 h\right) = \left(2k - \tilde{\nu} + \tfrac{5}{4}\right) i\pi h + o(h).$$

Writing $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, the real and imaginary part of the above condition read as

$$\tfrac{4}{3}\lambda_1 - \tfrac{1}{2}\, h \arg \lambda = \left(2k - \tilde{\nu} + \tfrac{5}{4}\right) \pi h + o(h), \tag{68}$$

$$-\tfrac{4}{3}\lambda_2 - \tfrac{1}{2}\, h \ln |\lambda| + \tfrac{1}{2}\, h \ln\left(\tfrac{\pi}{2}\tilde{\nu}^2 h\right) = o(h). \tag{69}$$

Now, we assume that $a < \lambda_1 < b$ and $\lambda_2 = o(1)$ as $h \to 0$. Then,

$$\arg \lambda = \arctan(\lambda_2/\lambda_1) = o(1), \quad \ln |\lambda| = \ln \lambda_1 + \tfrac{1}{2} \ln(1 + \lambda_2^2/\lambda_1^2) = \ln \lambda_1 + o(1).$$

Setting $\lambda_{k\tilde{\nu}} = \frac{3\pi}{16}(8k - 4\tilde{\nu} + 5)$, euqations (68) and (69) become

$$\begin{aligned} \lambda_1 &= \tfrac{3\pi}{16}(8k - 4\tilde{\nu} + 5)h + o(h) = \lambda_{k\tilde{\nu}} h + o(h), \\ \lambda_2 &= -\tfrac{3}{8}\left(h \ln \tfrac{1}{h} - h \ln \frac{\pi \tilde{\nu}^2}{2\lambda_{k\tilde{\nu}} h} \right) + o(h). \end{aligned}$$

□

Appendix A. Energy surfaces

Let $E \in \mathbb{C}$ and $\nu \in h(\mathbb{Z}+\frac{1}{2})$. The zeros of the function $\mathbb{C}^+ \to \mathbb{C}$, $r \mapsto (E-r^2)^2 - \nu^2/r^2$ are the roots of the sixth order polynomial $r^6 - 2Er^4 + E^2r^2 - \nu^2$ in r, which lie in the right half-plane $\mathbb{C}^+ = \{r \in \mathbb{C};\ \mathrm{Re}(r) > 0\}$. This polynomial has at most three different roots $r_0, r_1, r_2 \in \mathbb{C}^+$ in the right half-plane, whose squares are the roots $x_0, x_1, x_2 \in \mathbb{C}$ of the cubic polynomial $x^3 - 2Ex^2 + E^2x - \nu^2$ in x. If $E \in \mathbb{R}$, then an easy criterion for real-valuedness of the roots x_0, x_1, x_2 is the sign check of the polynomial discriminant

$$D_3 = (x_0 - x_1)^2 (x_0 - x_2)^2 (x_1 - x_2)^2 = \nu^2 (4E^3 - 27\nu^2).$$

The three roots are real if and only if $D_3 \geq 0$, that is iff $\nu^2 \leq 4E^3/27$. The roots are real and distinct, if and only if $\nu^2 < 4E^3/27$.

From Cardano's formula

$$\begin{aligned} x_0 &= 2E/3 + S_+ + S_-, \\ x_{1,2} &= 2E/3 - (S_+ + S_-)/2 \pm i\sqrt{3}\,(S_+ - S_-)/2 \end{aligned}$$

with $S_\pm = \sqrt[3]{-E^3/27 + \nu^2/4 \pm \sqrt{-D_3/108}}$ one learns $x_0 \to 0$ and $x_{1,2} \to E$ as $h \to 0$.

If one denotes by $\sqrt{E} \in \mathbb{C}^+$ the square root of $E \in \mathbb{C}$, which lies in the right half-plane, then $r_0 \to 0$ and $r_{1,2} \to \sqrt{E}$ as $h \to 0$, see also Figure 4.

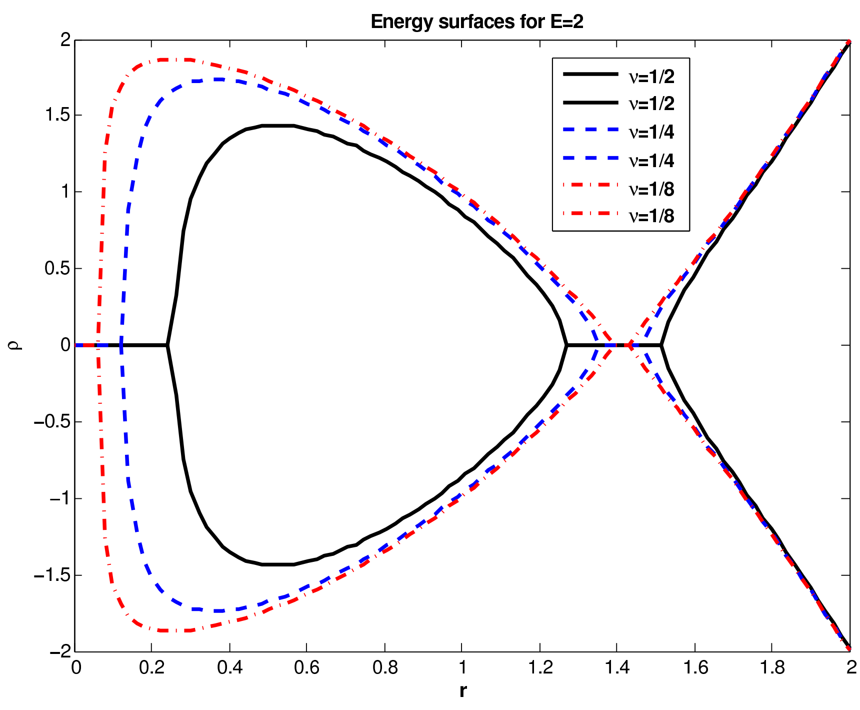

Figure 4. The energy surfaces $\{(r,\rho) \in \mathbb{R}^+ \times \mathbb{R};\ \rho^2 = (E-r^2)^2 - \nu^2/r^2\}$ for $E = 2$ and different values of $\nu \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$.

Appendix B. Spectrum of $P^+ = -h^2\Delta + |x|$

The Schrödinger operator P^+ has a locally bounded positive potential, which increases to infinity as $|x| \to \infty$. Hence, P^+ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and has purely

discrete spectrum. We are looking for eigenvalues $E \in]a,b[$ in a bounded positive interval $]a,b[\subset \mathbb{R}^+$. In polar coordinates $x = r(\cos\theta, \sin\theta)$, $r > 0$, $\theta \in \mathbb{T}$ the differential expression $-h^2\Delta_x + |x|$ reads as

$$-h^2\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + r.$$

Hence, E is a *formal* solution of the eigenvalue problem $(P^+ - E)\psi = 0$ if and only if there exists $l \in \mathbb{N}\cup\{0\}$ such that

$$-h^2\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l^2}{r^2}\right) w_l(r) + (r-E)w_l(r) = 0.$$

Substituting $w_l(r) = r^{-1/2}\, u_l(r)$, this is equivalent to

$$-h^2\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\frac{1}{4} - l^2}{r^2}\right) u_l(r) + (r-E)u_l(r) = 0. \tag{70}$$

The ordinary differential equation (70) has $r = 0$ as a regular singular point with exponents $\frac{1}{2} \pm l$, while $r = \infty$ is an irregular singular point of rank two. There are two linearly independent solutions, which behave as $r^{1/2}$ and $r^{1/2}\ln r$ (for $l = 0$) or $r^{1/2+l}$ and $r^{1/2-l}$ (for $l \neq 0$) near $r = 0$, see for example Chapter 5.5 in [28].

Looking at the irregular singularity at infinity, there is a fundamental system $u_\infty^\pm(r)$, which behaves as $(r-E)^{-1/4}\exp(\pm(r-E)^{3/2}/h)$ near $r = \infty$. Hence, a necessary condition for E being an eigenvalue of P^+ reads as follows.

Lemma B.1 *$E \in \sigma_{\mathrm{disc}}(P^+)$ if and only if there exists $l \in \mathbb{N}\cup\{0\}$ such that $\mathcal{W}(u^0, u_\infty^-) = 0$, where $\mathcal{W}(u^0, u_\infty^-) = u^0\,(u_\infty^-)' - (u^0)'\,u_\infty^-$ is the Wronskian of the two solutions to (70),*

$$\begin{aligned} u^0(r) &\sim r^{1/2+l}, & r \to 0,\\ u_\infty^-(r) &\sim (r-E)^{-1/4}\,\mathrm{e}^{-(r-E)^{3/2}/h}, & r \to \infty. \end{aligned}$$

Let us have a closer look at the potential of equation (70)

$$Q(r) = r - E + h^2(l^2 - 1/4)/r^2.$$

The origin $r = 0$ is a double pole and the coefficient $h^2(l^2 - 1/4)$ is positive for $l > 0$ and is negative for $l = 0$. This difference is crucial from the WKB point of view, since the level curves of $\mathrm{Re}(\int \sqrt{Q(r)}\,dr)$ in the complex r-plane are closed curves enclosing the origin for $l > 0$ and radial curves emanating the origin for $l = 0$. The WKB construction at a double pole for $l > 0$ has been treated in [14], and we here restrict ourselves to the application of this results.

We assume $l > 0$. For $h > 0$ sufficiently small, the potential Q has three distinct simple turning points: they are zeros of the cubic polynomial $r^3 - Er^2 + h^2(l^2 - 1/4)$. Two of them are at a distance $O(h)$ from $r = 0$, while the third is at a distance $O(h)$ from $r = E$. Since the polynomial discriminant

$$h^2(l^2 - \tfrac{1}{4})\left(\tfrac{1}{27}E^3 - \tfrac{h^2}{4}(l^2 - \tfrac{1}{4})\right) > 0$$

is positive, all three turning points are real. The first one is negative, while the other two are positive. We denote these two by $\alpha_1, \alpha_2 > 0$. The strategy for characterizing the quantization condition $\mathcal{W}(u^0, u_\infty^-) = 0$ is the following. We connect

(1) u^0 with two exact WKB solutions $\widetilde{u}^{\pm}$ built with the Langer-modified potential

$$\widetilde{Q}(r) = r - E + h^2 l^2/r^2$$

defined for $r \in]\alpha_1, \alpha_2[$ with the phase base point at α_1,

(2) $\widetilde{u}^{\pm}$ with two exact WKB solutions $u_l^{\pm}$ defined for $r \in]\alpha_1, \alpha_2[$ with the phase base point at α_2,

(3) $u_l^{\pm}$ with an exact WKB solution u_r^- defined for $r > \alpha_2$, which is collinear to u_∞^-.

We start with the connection near the origin. Proposition 11 in [14] proves existence of a non-zero constant $a = a(E, h) \neq 0$ such that

$$u^0 = a\,(1 + o(1))\,\widetilde{u}^+ + a\,(1 + o(1))\,\widetilde{u}^-$$

as $h \to 0$, where

$$\widetilde{u}^{\pm}(r) = \widetilde{Q}(r)^{-1/4} \exp\left(\pm\frac{1}{h}\int_{\widetilde{\alpha}_1}^{r} \widetilde{Q}(s)^{1/2} \mathrm{d}s\right) \widetilde{W}^{\pm}(r; r_l^{\pm}).$$

$\widetilde{\alpha}_1 > 0$ denotes the first positive turning point of $\widetilde{Q}$, while $r_l^{\pm} \in \mathbb{C}$ are two suitably chosen h-independent points in the complex plane such that $\mathrm{Re}(r_l^{\pm}) \in]\alpha_1, \alpha_2[$, $\pm\mathrm{Im}(r_l^{\pm}) > 0$, and $\widetilde{W}^{\pm}(r_l^{\pm}; r_l^{\pm}) = 1$.

Next, we connect near the turning point. The WKB solutions $u_l^{\pm}(r) = u^{\pm}(r; \alpha_2, r_l^{\pm})$ are of the form

$$u_l^{\pm}(r) = Q(r)^{-1/4} \exp\left(\pm\frac{1}{h}\int_{\alpha_2}^{r} Q(s)^{1/2} \mathrm{d}s\right) W^{\pm}(r; r_l^{\pm})$$

with $W^{\pm}(r_l^{\pm}; r_l^{\pm}) = 1$. Connecting $\widetilde{u}^{\pm}$ and $u_l^{\pm}$,

$$(\widetilde{u}^+, \widetilde{u}^-) = (u_l^+, u_l^-) \begin{pmatrix} \mathcal{W}(\widetilde{u}^+, u_l^-) & \mathcal{W}(\widetilde{u}^-, u_l^-) \\ -\mathcal{W}(\widetilde{u}^+, u_l^+) & -\mathcal{W}(\widetilde{u}^-, u_l^+) \end{pmatrix} \mathcal{W}(u_l^+, u_l^-)^{-1},$$

we evaluate the Wronskians, which are constant with respect to r, near the points $r = r_l^{\pm}$. By this choice we can exploit the fact that $\widetilde{Q}(r) = Q(r) + O(h^2)$ as $h \to 0$ uniformly in neighborhoods of $r = r_l^{\pm}$. Also the turning points differ by a term of order h^2, $\widetilde{\alpha}_1 = \alpha_1 + O(h^2)$, and one derives

$$\widetilde{u}^{\pm}(r) = u^{\pm}(r; \alpha_1, r_l^{\pm})\,(1 + O(h))$$

as $h \to 0$ uniformly in neighborhoods of $r = r_l^{\pm}$. Hence for the connection formula, we just have to account for solutions with different phase base points α_1, α_2. By Proposition 2.5 in [29], one has

$$\begin{aligned} \mathcal{W}(u_l^+, u_l^-) &= -\tfrac{2}{h}\,(1 + O(h)), \\ \mathcal{W}(\widetilde{u}^{\pm}, u_l^{\pm}) &= O(\mathrm{e}^{-\delta/h}), \\ \mathcal{W}(\widetilde{u}^{\pm}, u_l^{\mp}) &= \mp\tfrac{2}{h}\,\mathrm{e}^{\pm S_{12}/h}\,(1 + O(h)), \end{aligned}$$

where $\delta > 0$ does not depend on h, and

$$S_{12} = S_{12}(E, h) = \int_{\alpha_1}^{\alpha_2} Q(r)^{1/2}\,\mathrm{d}r$$

is the action integral between α_1 and α_2. We note, that the square root is taken such that S_{12} is purely imaginary with positive imaginary part. Collecting the previous formulae,

one obtains

$$(\widetilde{u}^+, \widetilde{u}^-) = (u_l^+, u_l^-) \begin{pmatrix} \mathrm{e}^{S_{12}/h}\,(1+O(h)) & O(\mathrm{e}^{-\delta/h}) \\ O(\mathrm{e}^{-\delta/h}) & \mathrm{e}^{-S_{12}/h}\,(1+O(h)) \end{pmatrix}$$

and

$$u^0 = a\,\mathrm{e}^{S_{12}/h}\,(1+o(1))\,u_l^+ + a\,\mathrm{e}^{-S_{12}/h}\,(\imath + o(1))\,u_l^- \qquad (h \to 0). \tag{71}$$

Finally, we connect to infinity. The solution $u_r^-(r) = u_r^-(r;\alpha_2, r_r)$ is defined as

$$u_r^-(r) = Q(r)^{-1/4} \exp\left(-\frac{1}{h}\int_{\alpha_2}^{r} Q(s)^{1/2}\mathrm{d}s\right) W^-(r;r_r)$$

with $r_r \in \mathbb{R}$ such that $r_r > \alpha_2$ and $W^-(r_r;r_r) = 1$. Since

$$Q(r) \sim r - E, \qquad r \to \infty,$$

u_r^- is collinear to u_∞^-, and $\mathcal{W}(u^0, u_\infty^-) = 0$ if and only if $\mathcal{W}(u^0, u_r^-) = 0$. By equation (71), it is enough to compute the Wronskians $\mathcal{W}(u_l^\pm, u_r^-)$. Choosing a branch cut for $Q(r)^{1/2}$ near α_2 as in Figure 2 of [14], one gets

$$\mathcal{W}(u_l^+, u_r^-) = -\tfrac{2}{h}\,(1+O(h))\,, \qquad \mathcal{W}(u_l^-, u_r^-) = \tfrac{2\imath}{h}\,(1+O(h))\,,$$

since $u_l^-(r) = -\imath\, u^+(r;\alpha_2, s_l^-)$, where s_l^- is the point r_l^- on the other Riemann surface. Hence, we have proven

Proposition B.2 *Assume $l > 0$. Then,*

$$\mathcal{W}(u^0, u_\infty^-) = 0 \quad\Longrightarrow\quad \mathrm{e}^{2S_{12}(E,h)/h} + 1 = o(1), \quad h \to 0,$$

with

$$S_{12}(E,h) = \int_{\alpha_1}^{\alpha_2} \sqrt{r - E + h^2(l^2 - \tfrac{1}{4})/r^2}\,\mathrm{d}r. \tag{72}$$

What remains, is a study of the asymptotic behavior of the action integral $S_{12}(E,h)$ as $h \to 0$. Substituting $r = Ey$ in (72), we rewrite

$$S_{12}(E,h) = \imath\, E^{3/2} I^+(\mu), \qquad \mu = \frac{\sqrt{l^2 - 1/4}}{E^{3/2}}\,h,$$

with

$$I^+(\mu) = \int_{y_1(\mu)}^{y_2(\mu)} \frac{\sqrt{y^2(1-y) - \mu^2}}{y}\,\mathrm{d}y,$$

where $y_1(\mu)$ and $y_2(\mu)$ are the first and second positive zero of the cubic polynomial $y^2(1-y) - \mu^2$. For $\mu > 0$ small enough, the polynomial has three zeros $y_0(\mu) < 0 < y_1(\mu) < y_2(\mu) < 1$ with $y_0(\mu), y_1(\mu) \to 0$ and $y_2(\mu) \to 1$ as $\mu \to 0$. When μ turns half around zero in the positive sense, that is when μ turns to $\mathrm{e}^{\imath\pi}\mu$, then $y_0(\mu)$ and $y_1(\mu)$ exchange their position, while $y_2(\mu)$ turns around 1. Hence, $I^+(\mathrm{e}^{\imath\pi}\mu) = I^+(\mu) + R^+(\mu) + T^+(\mu)$ with

$$R^+(\mu) = -\imath \int_{y_1(\mu)\frown y_0(\mu)} \frac{\sqrt{\mu^2 - y^2(1-y)}}{y}\,\mathrm{d}y$$

and $T^+(\mu) = \int_{\Gamma_1} \sqrt{y^2(1-y) - \mu^2}/y\,\mathrm{d}y = 0$, where Γ_1 is a contour around 1. Substituting $v = y\sqrt{1-y}$, we rewrite

$$R^+(\mu) = -\imath \int_{\mu\frown -\mu} \sqrt{\mu^2 - v^2}\,\frac{f(v)}{v}\,\mathrm{d}v,$$

where $f(v) = v^2/(v^2 - \frac{1}{2}y(v)^3)$ is holomorphic in a neighborhood of $v = 0$ and satisfies $f(v) = 1 + \frac{1}{2}v + f''(0)v^2 + O(v^3)$. Then,

$$R^+(\mu) = -\imath\,\mu \int_{1\curvearrowleft -1} \sqrt{1-w^2}\,\frac{f(\mu w)}{w}\,\mathrm{d}w = -\imath\,\mu\left(-\imath\pi - \tfrac{\pi}{4}\mu + O(\mu^3)\right)$$

as $\mu \to 0$, since $\int_{1\curvearrowleft -1}\sqrt{1-w^2}\,w^j\,\mathrm{d}w$ equals $-\imath\pi, -\frac{\pi}{2}, 0$ for $j = -1, 0, 1$, respectively. Hence,

$$I^+(\mathrm{e}^{\imath\pi}\mu) = I^+(\mu) - \pi\mu + \tfrac{\pi\imath}{4}\,\mu^2\left(1 + O(\mu^2)\right) \qquad (\mu \to 0),$$

and Proposition 8.2 yields

$$I^+(\mu) = \tfrac{2}{3} + \tfrac{\pi}{2}\mu + O(\mu^2|\ln\mu|) \qquad (\mu \to 0).$$

Remark B.3 With these asymptotics, the quantization condition of Proposition B.2 can be rephrased as follows. There exist an integer $k \in \mathbb{Z}$ and a natural number $l \in \mathbb{N}$ such that $S_{12}(E,h) = \imath\pi(k+\frac{1}{2})h + o(h)$ as $h \to 0$ or

$$E^{3/2} = \tfrac{3}{4}\pi\left(2k + 1 - \sqrt{l^2 - 1/4}\right)h + o(h) \qquad (h \to 0).$$

◇

The study of the resonant set of the full operator P and the spectrum of the upper level operator P^+ have revealed a parallel structure: A resonance E of P is characterized by a Bohr-Sommerfeld type condition

$$\tilde{t}\,\mathrm{e}^{2S_{01}(E,h)/h} + 1 = o(1) \qquad (h \to 0)\,,$$

where $\tilde{t}$ is related to non-adiabatic transitions, and $S_{01}(E,h)$ is an action integral. An eigenvalue λ of P^+ is characterized by a Bohr-Sommerfeld condition

$$\mathrm{e}^{2S_{12}(\lambda,h)/h} + 1 = o(1) \qquad (h \to 0)\,,$$

where $S_{12}(\lambda,h)$ is an action integral. The integrals can be expressed as

$$S_{01}(E,h) = \imath\,E^{3/2}I(\mu), \qquad \mu = h\,\tilde{\nu}\,E^{-3/2}, \qquad \tilde{\nu} \in \mathbb{N} + \tfrac{1}{2},$$
$$S_{12}(\lambda,h) = \imath\,\lambda^{3/2}I^+(\mu), \qquad \mu = h\sqrt{l^2 - \tfrac{1}{4}}\,\lambda^{-3/2}, \qquad l \in \mathbb{N},$$

where $I(\mu)$ and $I^+(\mu)$ share the same μ-asymptotics for $\mu \to 0$.

Appendix C. Symbols, frequency sets, microlocal solutions

Let us briefly recall the definition of the symbol classes used, as well as the notion of frequency set and microlocal solution.

Definition C.1 A smooth function $a(x,h)$ on an open set $\Omega \subset \mathbb{R}$ is called a $\mathcal{C}^\infty$-*symbol* of order $m \in \mathbb{Z}$, if there exist smooth functions $a_n(x) \in \mathcal{C}^\infty(\Omega,\mathbb{C})$ such that for all non-negative integers $\alpha \in \mathbb{N}$ and $N \in \mathbb{N}$,

$$\sup_{x\in\Omega}\,|\,\partial_x^\alpha(a(x,h) - \sum_{n=0}^{N} a_n(x)h^{n+m})\,| = \;O(h^{m+N+1}), \qquad h \to 0.$$

When this holds, we write $a(x,h) \sim \sum_{n=0}^\infty a_n(x)h^{n+m}$.

Definition C.2 An analytic function $a(x,h)$ on an open set $\Omega \subset \mathbb{C}$ is called a *Gevrey symbol* of index 2, if there exist functions $a_n(x)$ analytic in Ω such that for all compact subsets $K \subset \Omega$ and all large enough $C > 0$ there exists $\delta > 0$ with

$$\sup_{x \in K} \mid a(x,h) - \sum_{n=0}^{\frac{1}{C\sqrt{h}}} a_n(x) h^n \mid = O(e^{-\delta/\sqrt{h}}), \qquad h \to 0.$$

Definition C.3 Let $u \in \mathcal{S}'$ be a possibly h-dependent distribution and $(x_0, \xi_0) \in T^*\mathbb{R}$. Then $(x_0, \xi_0) \notin \mathrm{FS}(u)$ if there exists a Schwartz function χ_0 on phase space $T^*\mathbb{R}$ with $\chi_0(x_0, \xi_0) = 1$ such that for any $N \in \mathbb{N}$

$$\chi_0(x, hD_x)u = O(h^N), \qquad h \to 0.$$

$\mathrm{FS}(u)$ is called the *frequency set* of u, and u a *microlocal solution of* $u = 0$ near (x_0, ξ_0).

In Sections 7.3 and 7.4, we have used the following fact about the frequency set of WKB solutions:

Lemma C.4 *If $u(x,h) = a(x,h)\exp(i\phi(x)/h)$ where $\phi \in \mathcal{C}^1(\Omega, \mathbb{R})$ is a real phase and a is a $\mathcal{C}^\infty$-symbol, then $\mathrm{FS}(u) \subset \{(x,\xi);\ \xi = \partial_x\phi(x)\}$.*

Appendix D. Proof of Theorem 7.3

Lemma D.1 *Let $\tilde{\nu} \in \mathbb{N} - \frac{1}{2}$, $y \mapsto \psi(y)$ the function defined in (48) with $\psi(0) = \frac{E^{-3/4}}{\sqrt{2}}$, $E > 0$, and $v(y) = v(y,h)$ a solution of*

$$hD_y v(y) = \begin{pmatrix} y & h\,\tilde{\nu}\psi(y) \\ -h\,\tilde{\nu}\psi(y) & -y \end{pmatrix} v(y). \tag{73}$$

There exists a matrix-valued $\mathcal{C}^\infty$-symbol $M(y,h) = \mathrm{Id} + O(h)$, such that $w(y,h) = M(y,h)v(y,h)$ satisfies

$$\begin{pmatrix} hD_y - y & -\gamma \\ \overline{\gamma} & hD_y + y \end{pmatrix} w(y,h) = r(y,h)w(y,h), \tag{74}$$

where $\gamma = \frac{\tilde{\nu}}{\sqrt{2}}E^{-3/4}h + O(h^2)$ and $r(y,h) = O(h^\infty)$ uniformly in an interval around $y = 0$ together with all its derivatives.

Proof: We rewrite equation (73) as $hD_y v(y,h) = B(y,h)v(y,h)$ with

$$B(y,h) = B_0(y) + hB_1(y) = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} + h\begin{pmatrix} 0 & \tilde{\nu}\psi(y) \\ -\tilde{\nu}\psi(y) & 0 \end{pmatrix}$$

and equation (74) as $(hD_y - G(y,h))\,w(y,h) = r(y,h)w(y,h)$ with

$$G(y,h) \sim \sum_{n=0}^{\infty} G_n(y)h^n, \qquad G_0(y) = B_0(y), \qquad G_n(y) \equiv \begin{pmatrix} 0 & \gamma_n \\ -\overline{\gamma}_n & 0 \end{pmatrix} \qquad (n \geq 1).$$

We are looking for

$$M(y,h) \sim \sum_{n=0}^{\infty} M_n(y)h^n, \qquad M_n(y) = \begin{pmatrix} m_n(y) & q_n(y) \\ \overline{q}_n(y) & \overline{m}_n(y) \end{pmatrix} \qquad (n \geq 0)$$

such that

$$hD_yM = GM - MB$$

or equivalently for all $n \geq 0$

$$D_yM_{n-1} = \sum_{j=0}^{n} (G_jM_{n-j} - M_jB_{n-j}) \tag{75}$$

with convention $M_{-1} = 0$, $B_n = 0$ for $n \geq 2$.

Equation (75) is satisfied for $n = 0$, if we take $M_0(y) \equiv \mathrm{Id}$, i. e. $m_0(y) \equiv 1$ and $q_0(y) \equiv 0$.

Then, we have for $n \geq 1$

$$\begin{aligned} G_n &= -iM'_{n-1} - \sum_{j=0}^{n-1} G_jM_{n-j} + M_{n-1}B_1 + M_nB_0 \\ &= -iM'_{n-1} - \sum_{j=1}^{n-1} G_jM_{n-j} + M_{n-1}B_1 - 2y \begin{pmatrix} 0 & q_n \\ -\overline{q}_n & 0 \end{pmatrix}. \end{aligned}$$

Since $\psi(y) \in \mathbb{R}$ for $y \in \mathbb{R}$, the previous equation is equivalent to

$$\gamma_n = -iq'_{n-1}(y) - \sum_{j=1}^{n-1} \gamma_j \overline{m}_{n-j}(y) + \tilde{\nu}\psi(y)m_{n-1}(y) - 2yq_n(y), \tag{76}$$

$$0 = -im'_{n-1}(y) - \sum_{j=1}^{n-1} \gamma_j \overline{q}_{n-j}(y) - \tilde{\nu}\psi(y)q_{n-1}(y). \tag{77}$$

Let us start with $n = 1$. Substituting $y = 0$ in (76) yields

$$\gamma_1 = \tilde{\nu}\psi(0),$$

and we automatically obtain

$$q_1(y) = -\frac{1}{2y}\left(\gamma_1 - \tilde{\nu}\psi(y)\right),$$

which is smooth (even analytic) near $y = 0$. Setting $m_1(0) = 0$, equation (77) gives

$$m_1(y) = i\int_0^y \left(\gamma_1\overline{q}_1(y') + \tilde{\nu}\psi(y')q_1(y')\right) dy'.$$

If γ_j, m_j, q_j with $m_j(0) = 0$ are determined for $1 \leq j \leq n$, then (76) yields

$$\gamma_{n+1} = -iq'_n(0)$$

and

$$q_{n+1}(y) = -\frac{1}{2y}\big(\gamma_n + iq'_n(y) + \sum_{j=1}^{n} \gamma_j\overline{m}_{n+1-j}(y) - \tilde{\nu}\psi(y)m_n(y)\big),$$

which is smooth (even analytic) near $y = 0$. Setting $m_{n+1}(0) = 0$, equation (77) then gives

$$m_{n+1}(y) = i\int_0^y \big(\sum_{j=1}^{n+1} \gamma_j\overline{q}_{n+2-j}(y') + \tilde{\nu}\psi(y')q_{n+1}(y')\big)dy'.$$

Since $m_n(y)$ and $q_n(y)$, $n \geq 0$, are smooth functions near $y = 0$, there exists a matrix-valued $\mathcal{C}^\infty$-symbol $M(y,h)$ with $M(y,h) \sim \sum_{n\geq 0} M_n(y)h^n$ such that (74) holds. □

Lemma D.2 *Let $\kappa_{\frac{\pi}{4}}(y,\eta) = \frac{1}{\sqrt{2}}(y-\eta, y+\eta)$ be the $\frac{\pi}{4}$-rotation in phase space $T^*\mathbb{R}$. The metaplectic operator V of the transpose linear canonical transformation $\kappa^*_{\frac{\pi}{4}} = \kappa_{-\frac{\pi}{4}}$ satisfies*

$$V(hD_y - y) = -\sqrt{2}yV\,, \qquad V(hD_y + y) = \sqrt{2}\,hD_yV.$$

Moreover, $\mathrm{FS}(Vu) = \kappa_{-\frac{\pi}{4}}\mathrm{FS}(u)$ *for* $u \in \mathcal{S}'$.

Proof: One applies Theorem 2.15 in [12] for the linear canonical transformation $\kappa_{\frac{\pi}{4}}$. □

Remark D.3 A formula for V in terms of oscillatory integrals is given by

$$Vg(y) = e^{i\pi/8}(\sqrt{2}\pi h)^{-1/2}\int_{\mathbb{R}} e^{-\frac{i}{2h}(y^2-2\sqrt{2}xy+x^2)}g(x)dx,$$

see Theorem 4.53 in [12] or Proposition 5.3 in [29]. ◇

Appendix E. Gevrey symbols

Lemma E.1 *The analytic functions $m_0(x) \equiv 1$, $q_0(x) \equiv 0$,*

$$\begin{aligned} q_n(x) &= -\frac{1}{2x}\Big(i\int_0^x q''_{n-1}(t)dt - i\sum_{j=2}^{n-1} q'_{j-1}(0)\,\overline{m}_{n-j}(x) \\ &\qquad -m_{n-1}(x)\,\tilde{\nu}\,\psi(x) + \overline{m}_{n-1}(x)\,\tilde{\nu}\psi(0)\Big), \qquad n \geq 1, \\ m_n(x) &= i\,\tilde{\nu}\int_0^x q_n(t)\psi(t)dt + i\sum_{j=1}^{n}\gamma_j\int_0^x \overline{q}_{n+1-j}(t)dt, \qquad n \geq 1 \end{aligned}$$

with $\gamma_1 = \tilde{\nu}\psi(0)$ and $\gamma_n = -iq'_{n-1}(0)$ for $n \geq 2$, which have been introduced in the proof of Lemma D.1, have analytic resummations $q(x,h)$ and $m(x,h)$ in some neighborhood of $x = 0$, which are Gevrey symbols of index 2.

Proof: Let $\Omega = \{x \in \mathbb{C};\, |x| < r\}$ and $\Omega_t = \{x \in \mathbb{C};\, |x| \leq r-t\}$ for $0 < t \leq r$. It is enough to show, that there exist positive constants $D_m, D_q, C > 0$ such that for all $n \in \mathbb{N}$

$$\sup_{x\in\Omega_t}|q_n(x)| \leq D_qC^n\frac{(2n)^{2n}}{t^{2n}}, \qquad \sup_{x\in\Omega_t}|m_n(x)| \leq D_mC^n\frac{(2n)^{2n}}{t^{2n}}. \tag{78}$$

We use the following bounds: if $f(x)$ is a holomorphic function in Ω, which satisfies for some positive constant $M > 0$

$$\sup_{x\in\Omega_t}|f(x)| \leq \frac{M}{t^k},$$

then

$$\sup_{x\in\Omega_t}\left|\int_0^x f(u)du\right| \leq \frac{M}{(k-1)t^{k-1}}, \qquad \sup_{x\in\Omega_t}\left|\frac{1}{x}\int_0^x f(u)du\right| \leq \frac{M}{t^k},$$

and

$$\sup_{x\in\Omega_t}|f'(x)|,\ \sup_{x\in\Omega_t}\left|\tfrac{f(x)}{x}\right| \leq \frac{(k+1)^{k+1}M}{k^k\,t^{k+1}}, \qquad \sup_{x\in\Omega_t}|f''(x)| \leq \frac{(k+2)^{k+2}M}{k^k\,t^{k+2}}.$$

We prove (78) by induction:

$$
\begin{aligned}
\sup_{x\in\Omega_t}|q_n(x)| &\leq \sup_{x\in\Omega_t}\left|q''_{n-1}(x)\right| + \sum_{j=2}^{n-1}|q'_{j-1}(0)|\sup_{x\in\Omega_t}\left|\tfrac{m_{n-j}(x)}{x}\right| \\
&\qquad\qquad +2|\tilde\nu|\sup_{x\in\Omega_t}\left|\tfrac{m_{n-1}(x)}{x}\right|\sup_{x\in\Omega}|\psi(x)| \\
&\leq D_qC^{n-1}\frac{(2n)^{2n}}{t^{2n}} + \sum_{j=2}^{n-1}D_qC^{j-1}\frac{(2j-1)^{2j-1}}{t^{2j-1}}D_mC^{n-j}\frac{(2n-2j+1)^{2n-2j+1}}{t^{2n-2j+1}} \\
&\qquad\qquad +2|\tilde\nu|D_mC^{n-1}\frac{(2n-1)^{2n-1}}{t^{2n-1}}\sup_{x\in\Omega}|\psi(x)| \\
&\leq D_q\left(1+D_m+2|\tilde\nu|\,r\,\frac{D_m}{D_q}\sup_{x\in\Omega}|\psi(x)|\right)C^{n-1}\frac{(2n)^{2n}}{t^{2n}},
\end{aligned}
$$

where we used the inequality

$$
k^k(N-k)^{N-k}\leq\left(\tfrac{N}{2}\right)^N \qquad (0<k<N).
$$

Taking C large enough so that $C\geq 1+D_m+2r|\tilde\nu|\frac{D_m}{D_q}\sup_{x\in\Omega}|\psi(x)|$, we obtain the first bound in (78). Similarly, one has

$$
\begin{aligned}
\sup_{x\in\Omega_t}|m_n(x)| &\leq |\tilde\nu|\sup_{x\in\Omega}|\psi(x)|D_qC^n\frac{(2n)^{2n}}{(2n-1)t^{2n-1}} \\
&\quad+\sum_{j=1}^{n}D_qC^{j-1}(2j-1)^{2j-1}D_qC^{n+1-j}\frac{(2n+2-2j)^{2n+2-2j}}{(2n+1-2j)\,t^{2n}} \\
&\leq \left(|\tilde\nu|\,r\sup_{x\in\Omega}|\psi(x)|+2D_q\right)D_qC^n\frac{(2n)^{2n}}{t^{2n}}.
\end{aligned}
$$

Taking D_m so that $D_m\geq(|\tilde\nu|r\sup_{x\in\Omega}|\psi(x)|+2D_q)D_q$, we get the second bound in (78). $\square$

Appendix F. Proof of Lemma 7.9

In this appendix, we prove one of the two formulae of Lemma 7.9, i.e. compute the asymptotic exapnsion of $\tilde f^+$ microlocally near σ_r^-.

Recall that near $y=0$

$$
\tilde f^+(\phi^{-1}(y)) = M(y,h)^{-1}V^{-1}f^+(y)
$$

where the inverse of the metaplectic operator is given by

$$
V^{-1}f^+(y) = e^{i\pi/8}(\sqrt2\pi h)^{-1/2}\int_{\mathbb R}e^{i(y^2+z^2-2\sqrt2yz)/(2h)}f^+(z)dz,
$$

and that

$$
f^+={}^t(f_1^+,f_2^+),\quad f_1^+=-\frac{\gamma}{\sqrt{2}y}\chi_{(0,\infty)}(y)\,|y|^{\frac{i}{2h}|\gamma|^2},\quad f_2^+=\chi_{(0,\infty)}(y)\,|y|^{\frac{i}{2h}|\gamma|^2}.
$$

Let us compute the asymptotic expansion of

$$
V^{-1}f_2^+(y)=e^{i\pi/8}(\sqrt2\pi h)^{-1/2}\int_0^\infty e^{i\varphi(y,z)/h}dz,\quad \varphi(y,z)=\frac12(y^2+z^2-2\sqrt2yz+|\gamma|^2\ln z).
$$

Since $\gamma = O(h)$, the phase function $\varphi(y,z)$ has two real critical points

$$z_c^{\pm}(y) = (y \pm \sqrt{y^2 - |\gamma|^2})/\sqrt{2},$$

which are the roots of $\frac{\partial\varphi}{\partial z}(y,z) = z - \sqrt{2}y + \frac{|\gamma|^2}{2z}$. The phase of the asymptotic expansion will be given by $\varphi(y, z_c^{\pm}(y))$. Since we are on σ_r^-, we can assume $x > \sqrt{E}$, i.e. $y > 0$ and independently from h. Then,

$$z_c^+(y) = \sqrt{2}y + O(h^2), \quad z_c^-(y) = \frac{|\gamma|^2}{2\sqrt{2}y} + O(h^2)$$

and

$$\varphi(y, z_c^{\pm}(y)) = \mp\frac{1}{2}y^2 + O(h^2).$$

In view of Lemma C.4, this means that the critical points $z_c^+(y)$ and $z_c^-(y)$ will contribute on σ_r^- and σ_r^+, respectively. Hence, we have only to compute the contribution from $z_c^+(y)$. Since $\frac{\partial^2\varphi}{\partial z^2}(y, z_c^+(y)) = 1 - \frac{|\gamma|^2}{2z_c^+(y)^2} = 1 + O(h^2)$, $z_c^+(y)$ is a non-degenerate critical point, and the stationary phase theorem (e.g. Proposition 5.2 in [9]) says that

$$V^{-1}f_2^+(y) = e^{i\pi/8}\, 2^{1/4}\, e^{i\varphi(y,z_c^+(y))/h} + O(h)$$

microlocally near σ_r^-. Comparing $i\varphi(\phi(x), z_c^+(\phi(x))) = -\frac{i}{2}\phi(x)^2 + O(h^2)$ with $z(x;r_2) = \int_{r_2}^x \sqrt{\nu^2 - t^2(E-t^2)^2}/t\,dt$, we observe for $x > r_2$

$$\begin{aligned}
-\frac{d}{dx}z(x;r_2) &= -\sqrt{\nu^2 - x^2(E-x^2)^2}/x = -i(x^2 - E) + O(h^2) \\
&= \frac{d}{dx}\left(-\tfrac{i}{2}\phi(x)^2\right) + O(h^2),
\end{aligned}$$

since $\nu^2 = O(h^2)$ and $\phi(x)\phi'(x) = x^2 - E$. Because of $z(r_2;r_2) = 0$, one obtains

$$-z(x;r_2) = -\tfrac{i}{2}\phi(x)^2 - \tfrac{i}{2}\phi(r_2)^2 + O(h^2).$$

If $\phi(r_2)^2 = O(h^2)$, then $i\varphi(\phi(x), z_c^+(\phi(x))) = -z(x;r_2) + O(h^2)$. Consequently,

$$V^{-1}f_2^+(\phi(x)) = e^{i\pi/8}\, 2^{1/4}\, e^{-z(x;r_2)/h}\,(1 + O(h))$$

and

$$\tilde{f}^+(x) = e^{i\pi/8}\, 2^{1/4}\, e^{-z(x;r_2)/h}\begin{pmatrix}0\\1\end{pmatrix}(1 + O(h))$$

microlocally near σ_r^-, since $M(\phi(x),h) = \mathrm{Id} + O(h)$.

It remains to prove $\phi(r_2)^2 = O(h^2)$. The turning point r_2 is a zero of the function $x \mapsto \nu^2/x^2 - (E-x^2)^2$. This is equivalent to $\det(a(r_2,0)) = 0$ with

$$a(x,\xi) = \begin{pmatrix}-\xi + x^2 - E & \frac{\nu}{x}\\ -\frac{\nu}{x} & -\xi - x^2 + E\end{pmatrix}$$

the symbol of the operator $A(x) - hD_x$. The first step of the normal form transformation of Theorem 7.3 reads on the symbol level as

$$a(x,\xi) = \phi'(x)\tilde{a}(\phi(x), \xi/\phi'(x))$$

with

$$\tilde{a}(x,\xi) = \begin{pmatrix}-\xi + x & \nu\psi(x)\\ -\nu\psi(x) & -\xi - x\end{pmatrix}.$$

Since $\phi'(r_2) \neq 0$, one has $\det(a(r_2,0)) = 0$ if and only if $\det(\tilde{a}(\phi(r_2),0)) = 0$. The rest of the normal form transformation is

$$\sqrt{2}\, q \circ \kappa_{\frac{\pi}{4}} = M \sharp_h \tilde{a} \sharp_h M^{-1}$$

with

$$q(y,\eta) = \begin{pmatrix} y & \frac{\gamma}{\sqrt{2}} \\ -\frac{\overline{\gamma}}{\sqrt{2}} & -\eta \end{pmatrix}$$

the symbol of the normal form operator Q, and $\sharp_h$ the Moyal product of semiclassical Weyl calculus. The h-asymptotics of $\sharp_h$ (e.g. Proposition 7.7 in [9]) together with the linear ξ-dependance of $\tilde{a}(x,\xi)$ yield

$$M \sharp_h \tilde{a} \sharp_h M^{-1} = M\tilde{a}M^{-1} - ihM'M^{-1} = M\tilde{a}M^{-1} + O(h^2).$$

Hence, $\tilde{a} = \sqrt{2}\, M^{-1}(q \circ \kappa_{\frac{\pi}{4}})M + O(h^2)$ and

$$\det(a(r_2,0)) = 0 \;\Leftrightarrow\; \det(q \circ \kappa_{\frac{\pi}{4}}(r_2,0)) = O(h^2) \;\Leftrightarrow\; \phi(r_2)^2 = O(h^2).$$

References

[1] J. Aguilar and J.M. Combes. A class of analytic pertubations for one body Schrödinger Hamiltonians *Commun. Math. Phys.*, 22: 269–279, 1971.

[2] J. Avron and A. Gordon. Born-Oppenheimer approximation near level crossing. *Phys. Rev. Lett.*, 85(1): 34–37, 2000.

[3] J. Avron and A. Gordon. Born-Oppenheimer wave function near level crossing. *Phys. Rev. A*, 62: 062504-1–062504-9, 2000.

[4] H. Baklouti. Asymptotique des largeurs de résonances pour un modèle d'effet tunnel microlocal. *Ann. Inst. Henri Poincaré, Phys. Théor.* 68(2): 179–228, 1998.

[5] L. Cederbaum, R. Friedman, V. Ryaboy, and N. Moiseyev. Conical intersections and bound molecular states embedded in the continuum. *Phys. Rev. Lett.*, 90(1): 013001-1–013001-4, 2003.

[6] M. S. Child. *Semiclassical Mechanics with Molecular Applications*. Oxford, 1991.

[7] Y. Colin de Verdière and B. Parisse. Équilibre instable en régime semi-classique. I. Concentration microlocale. *Comm. Partial Differential Equations*, 19(9&10): 1535–1563, 1994.

[8] J.M. Combes, P. Duclos, and R. Seiler. The Born Oppenheimer approximation. *Rigorous Atomic and molecular Physics*, 185–212, 1981.

[9] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*. Cambridge University Press, Lecture Note Series 268, 1999.

[10] W. Domcke, D. Yarkony, and H. Köppel (eds.). *Conical intersections*. World Scientific Publishing, Advanced Series in Physical Chemistry, Vol. 15, 2004.

[11] C. Fermanian-Kammerer and P. Gérard. Mesures semi-classiques et croisements de mode. *Bull. S.M.F*, 130(1): 123–168, 2002.

[12] G. Folland. *Harmonic analysis in phase space*. Annals of mathematics studies, 122, Princeton University Press, Princeton, 1989.

[13] S. Fujiié and T. Ramond Matrice de scattering et résonances associées à une orbite hétérocline. *Ann. Inst. Henri Poincaré, Phys. Théor.*, 69(1): 31–82, 1998.

[14] S. Fujiié and T. Ramond. Exact WKB analysis and the Langer modification with application to barrier top resonances. C. Howls (ed.), Toward the exact WKB analysis of differential equations, linear or non-linear, Kyoto University Press, 15–31, 2000.

[15] C. Gérard and A. Grigis. Precise estimates of tunneling and eigenvalues near a potential barrier. *J. Differ. Equations*, 72(1): 149-177, 1988.

[16] G.A. Hagedorn. A time dependent Born-Oppenheimer approximation. *Commun. Math. Phys.*, 77: 1–19, 1980.

[17] G.A. Hagedorn. Molecular propagation through electron energy level crossings. *Memoirs A.M.S. 536*, 111, 1994.

[18] G.A. Hagedorn and A. Joye. A time-dependent Born-Oppenheimer approximation with exponentially small error estimates. *Commun. Math. Phys.*, 223(3): 583–626, 2001.

[19] B. Helffer and J. Sjöstrand. Semiclassical analysis for Harper's equation III. *Mém. Soc. Math. France (N.S.)*, 39, 1989.
[20] I. Herbst. Dilation analyticity in constant electric field I. The two body problem. *Commun. Math. Phys.* 64:279–298, 1979.
[21] T. Kato. *Perturbation theory for linear operators*, corr. print. of the 2nd ed. Classics in mathematics, Springer Verlag, Berlin, 1995.
[22] M. Klein, A. Martinez, R. Seiler, and X.P. Wang. On the Born-Oppenheimer expansion for polyatomic molecules. *Comm. Math. Phys.*, 143: 607–639, 1992.
[23] C. Lasser and C. Fermanian-Kammerer. Wigner measures and codimension two crossings. *J. Math. Phys.*, 44(2): 507–527, 2003.
[24] C. Lasser and S. Teufel. Propagation through conical crossings: an asymptotic semigroup. *Commun. Pure Appl. Math.* 58: 1188–1230, 2005.
[25] A. Martinez. Estimates on complex interactions in phase space. *Math. Nachr.* 167: 203–254, 1994.
[26] S. Nakamura. On an example of phase space tunneling. *Ann. Inst. Henri Poincaré, Phys. Théor.* 63(2): 211–229, 1995.
[27] L. Nédélec. Résonances semi-classique pour l'opérateur de Schrödinger matriciel en dimension deux. *Ann. Inst. Henri Poincaré Phys. Théor.*, 65(2): 129–162, 1996.
[28] F. Olver. Asymptotics and special functions. Academic Press, New York, 1974.
[29] T. Ramond. Semiclassical study of quantum scattering on the line. *Commun. Math. Phys.*, 177: 221–254, 1996.
[30] F. Rellich. Störungstheorie der Spektralzerlegung. I. *Math. Ann.*, 113: 600-619, 1937.
[31] J. Sjöstrand and M. Zworski. Complex scaling and the distribution of scattering poles. *J. Amer. Math. Soc.*, 4: 729–769, 1991.
[32] H. Spohn and S. Teufel. Adiabatic decoupling and time-dependent Born-Oppenheimer theory. *Commun. Math. Phys.*, 224(1): 113–132, 2001.
[33] D. Yarkony. Conical interesections: the new conventional wisdom. *J. Phys. Chem. A*, 105: 6277–6293, 2001.
[34] M. Zworski. Resonances in physics and geometry. *Notices Amer. Math. Soc.*, 46(3), 1999.

(S. Fujiié) Mathematical Institute, Tohoku University, Aoba, Aramaki-aza, Aoba-ku, Sendai, 980-8578 Japan
E-mail address: fujiie@math.tohoku.ac.jp

(C. Lasser) Fachbereich Mathematik, Freie Universität Berlin, D-14195 Berlin, Germany
E-mail address: lasser@math.fu-berlin.de

(L. Nédélec) L.A.G.A., Institut Galilée, Université de Paris Nord, av. J.B. Clement, F-93430 Villetaneuse, France IUFM de l'academie de Rouen, France
E-mail address: nedelec@math.univ-paris13.fr